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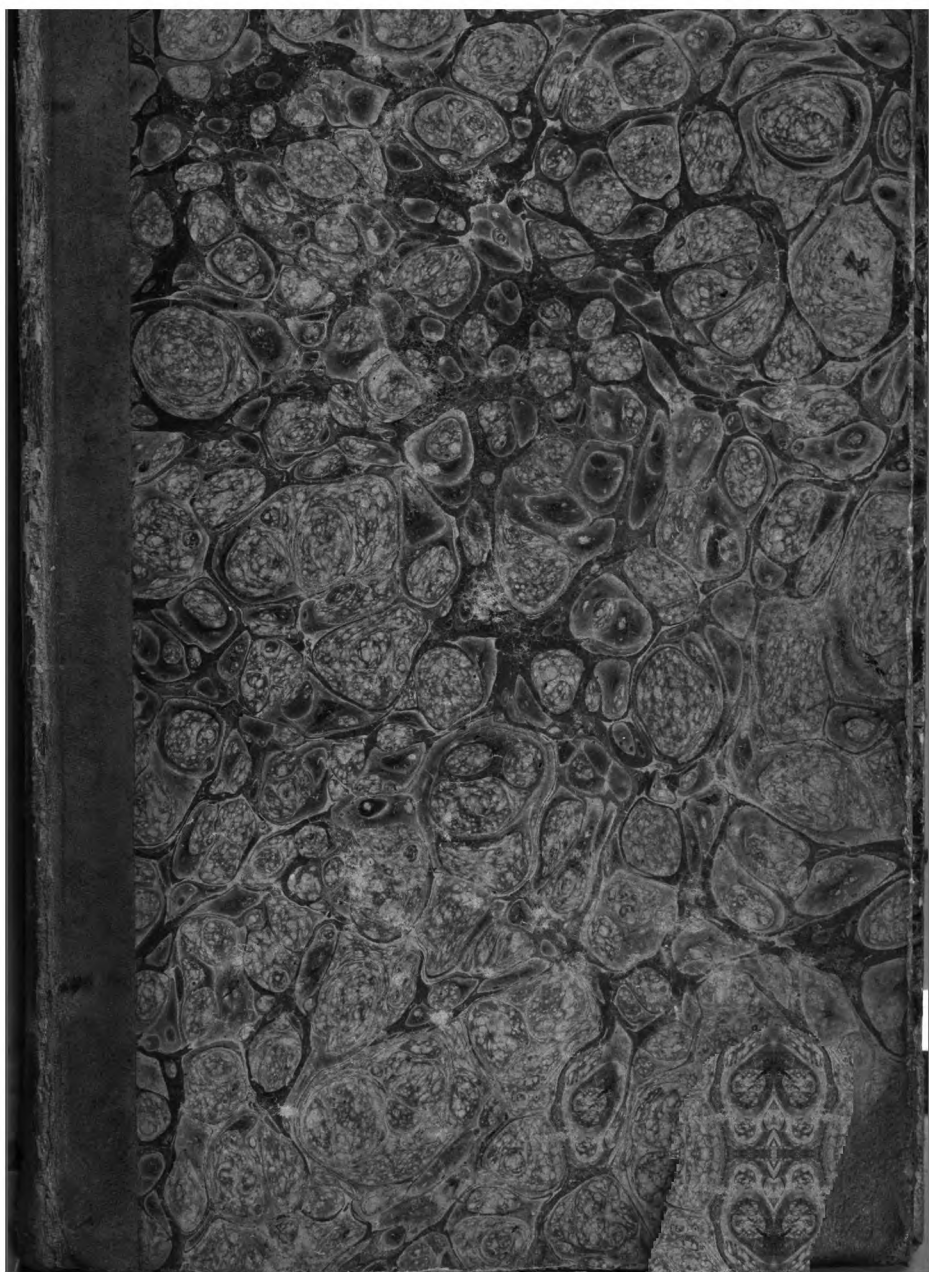
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*William B. Rogers.*

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AN INTRODUCTION  
TO THE  
MATHEMATICAL PRINCIPLES  
OF  
NATURAL PHILOSOPHY,  
ADAPTED TO THE USE OF BEGINNERS; AND ARRANGED MORE  
PARTICULARLY FOR THE CONVENIENCE OF THE  
JUNIOR STUDENTS  
OF  
WILLIAM & MARY COLLEGE, VIRGINIA.

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By PATRICK KERR ROGERS, M. D.  
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## PART I.

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# DYNAMICS.

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Dynamics is that branch of Natural Philosophy, which treats of the general or abstract laws of motion and moving forces. In investigating the various conditions of motion, the consideration of mass, space, and time, is necessarily involved.

### SECTION 1.

#### I. DEFINITIONS.

1. *Mass* or *body* denotes some determinate quantity of matter. Continuity of matter, or a connection of parts is implied, when we speak of a mass or body.

2. A *particle*, in mechanical inquiries, means one of the smallest integrant parts of a body, which, if separated from the rest, retains its mechanical properties unchanged.

3. The *density* of a body, is its quantity of matter compared with the quantity of matter in another body of equal magnitude. The density of a body is estimated by its weight or gravity, when the magnitude is given.

4. *Universal space* is the unbounded expanse in which all the bodies of the universe exist, move, and operate. All the parts of universal space are necessarily conceived to be immoveable.

5. *Particular space* is a distance limited by two points, a surface limited by lines, or a place limited by one or several surfaces.

6. *Absolute space* is some magnitude of universal space fixed in position.

7. Relative space is some magnitude changing its position in universal space. Thus, the space occupied by the earth, which is always changing its position, is *relative*.

8. An *instant* is to time what a point is to linear space. The continual succession of instants generates time ; and the interval between any two instants is a *finite* portion of time. Our abstract idea of time depends upon the memory of past sensations. Yet it is obvious that the results of an intellectual measure of time would be liable to the greatest uncertainty. We require, however, some measure of equal times to which we may appeal with confidence. Such a measure we derive from the changes of position in external objects, which by long and attentive observation appear to return to similar positions at equal intervals. Thus, we obtain from astronomical observations the well known measures of time, implied by the terms day, hour, minute and second.

9. Physical motion is a progressive change of position in some space which is either quiescent or conceived to be so. It has been distinguished into Absolute and Relative.

10. Absolute motion is a progressive change of position with respect to some absolute space.

11. Relative motion is a progressive change of position with respect to some relative space.

12. When a body or point passes through equal spaces in equal portions of time, the motion is said to be *uniform* or *equable*.

13. When a body passes over *greater* or *less* spaces in equal successive intervals of time, it is a *varying* motion. In the former case the motion is said to be *accelerated*, in the latter *retarded*.

14. The *rate* of motion is expressed by the term *velocity*. The length of linear space described uniformly in any unit of time, as a second or an hour, is the measure of its velocity.

15. *Momentum* and *impetus* are terms employed to express the quantity of motion.





Quantity of motion is very different from simple velocity. If three equal bodies describe equal spaces in a given time, their velocities will be equal, and so will be their momenta or quantities of motion. Unite two of these bodies into one, and let the single and double body pass over equal spaces in a given time. The velocity of the single body is still equal to that of the double body, but each constituent of the latter has a quantity of motion equal to that of the former.

16. Any principle conceived to be the immediate agent which changes the condition of bodies may be termed a Power.

Thus, when magnets attract or repel, the immediate agent which produces the motion, whether it be the iron itself or some distinct element inherent in it, may be termed a power, and is sometimes when in action denominated the *efficient cause*.

The word cause, however, even in physical enquiries, has a less determinate, yet a more extended signification. Power in the above sense is of continual existence; a cause exists only when effects or changes are produced. The power by which a stone, abandoned in the air, is made to descend, is constantly present in the stone and the earth, but stones are not always descending. A horse in a mill is the power which gives motion to the machinery; if he stand still, the power is nevertheless present, but it is only when he moves, that he *causes* the mill to move; when he ceases to exert his *power*, his quiescence is the *cause* of quiescence in the machinery. Even the state of things in one moment is considered as the *cause* of the state of things in the next moment.

17. *Force* is the exertion of some power in a determinate quantity which changes or tends to change the state of motion or rest in bodies; and is of two kinds, *Impulse* and *Pressure*.

18. *Impulse* is an instantaneous force, an exertion of momentary duration; as, when a hammer strikes an anvil.

19. *Pressure* is a force of sensible duration. Of this kind are attractions and repulsions acting at sensible distances, as well as pressures properly so called, which take place in bodies apparently in contact.

20. A pressure is said to be constant, uniform or unvarying, when its intensity remains always the same; that is, when it generates or destroys equal quantities of motion in equal times.

**21. Similar points of space are those which divide given portions of linear space in the same ratio.**

Thus, the points, which divide a line of one foot in length into 4 inches and 8 inches, and a line of one yard in length, into 12 inches and 24 inches, are similar.

**22. Similar instants of time are those which divide given portions of time in the same ratio.**

**23. Motions are similar, when the velocities in similar instants of time or in similar points of space are in a constant ratio.**

Thus, if the times in which two equal spaces are described are one minute and one hour, whatever may be the ratio of the velocities in the first instant, will be their ratio in any similar instant; as the end of the 10th second in one of the times given, and the end of the 10th minute, in the other time given, if the motions are similar.

And if the spaces described in equal times be 1 foot and one yard, 4 inches of the less space will be described, while 12 inches of the greater are described, if the motions are similar.

**24. Similar forces are those which produce similar motions. Hence,**

*First*, all impulses are similar forces, however they may differ in intensity, because they produce uniform motions.

*Second*, all uniform pressures are similar forces.

*Third*, all varying pressures, whose intensities in similar instants of time, or in similar points of space, are in a constant ratio, are similar forces. These are said to vary according to the same law.

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## II. LAWS OF MOTION OR GENERAL RULES ACCORDING TO WHICH FORCES ACT.

**25. FIRST LAW.** *A body continues in a state of rest, or proceeds in uniform rectilinear motion, unless affected by some mechanical force.*





This rule, which has been called the *law of inertia*, contains three distinct propositions :

**26. First proposition.** A body in a state of rest continues at rest unless affected by some mechanical force.

Thus, when a vessel filled with water, is drawn suddenly along the floor, the water dashes over its posterior side. The water, persevering in its state of rest, is left behind. In the same manner, when a coach is dragged forwards, the persons in it strike back against its posterior part. But, properly speaking, it is the coach that strikes on them on account of their perseverance in a state of rest, while the coach is dragged forward.

**27. Second proposition.** A body in motion proceeds in a right line unless caused to deviate from it by some mechanical force.

For, if it deviate from that line, it must move in some other direction, and it is inconceivable how it should deviate towards any one side more than another, and still more how a change of direction should take place without any changing force.

Rectilinear motion is indeed never known to take place among the great bodies of the universe, nor is it possible for us in our experiments to produce it. Nevertheless the proposition is undoubtedly true in theory; and in our experiments, in proportion as we remove the influence of changing forces, the direction of the motion becomes the more nearly rectilinear.

**28. Third proposition.** A body in motion proceeds *uniformly*, passing through equal spaces in equal times, unless affected by some mechanical force.

Every change in the state of a body implies a changing cause. If the motion be not uniform, it must at some point be diminished or increased. In either case, a change arises which we cannot but ascribe to some natural cause; and we say a mechanical force affects the moving body.

**29. SECOND LAW.** Every change of motion is proportional to the force impressed, and is made in the direction of that force.

This rule, which has been called the *law of several forces*, contains two distinct propositions :

**30. First proposition.** Every change of motion is *proportional* to the force impressed.

As a change of motion is the effect, and a changing force the cause, the former may be safely assumed as the measure of the latter. Thus, if any motion is doubled by the application of a changing force, this force is evidently equal to the original moving force; and if one half of any motion be destroyed, the changing force is in this case half as great as the original moving force.

**31. Second proposition.** Every change of motion is made in the *direction* of the force impressed.

As the direction of any motion is the natural indication of the direction of the moving force, so the direction of any change in a motion already existing is the indication of the direction of the changing force.

**32. THIRD LAW.** *Action and re-action are always equal, and in opposite directions.*

This rule, which may be called the *law of mutual action*, also contains two distinct propositions :

**33. First proposition.** The reciprocal actions of bodies are always equal; or, the forces which bodies exert on one another are always of equal intensity, whether the bodies attract or repel, whether they operate by impulse or pressure.

Whatever body draws, presses, or impels another, is as much drawn, pressed, or impelled, by that other.

**Fig. 1.** Let A and B be two equal free bodies which attract: experience shows that they will approach and meet with equal velocity. Consequently, the action of A upon B is equal to the action of B upon A.

Again; let A be greater than B. Their mutual attractions will be greater than in the former case; yet each body will be equally affected by them. For, each particle of A is attracted by all the particles of B, and each particle of B is attracted by all the particles of A. Therefore, if  $m$  denote the number of particles in A, and  $n$  the number of particles in B;  $n \times m$  will express the attraction which A exerts upon B, and  $m \times n$  will express the attraction which B exerts upon A. Consequently, the attraction of A to B is equal to the attraction of B to A. The same kind of reasoning will apply, if A and B exert a mutual repulsion.





It is to be understood, that from whatever side we estimate the quantity of the force, it is only a single though a mutual force.

34. *Second proposition.* Mutual forces are always exerted in opposite directions.

*Fig. 1.* Let A be immoveable and B free ; when in this state A attracts B, the effect on B must be a motion in the direction of the line from B to A : and let B be immoveable and A free ; the effect on A will be a motion in the direction of the line from A to B. And it is inconceivable, that the motion should be otherwise, when both bodies are free. The same kind of reasoning will apply, if A and B exert a mutual repulsion.



### III. FUNDAMENTAL ANALOGIES.

#### PROP. 1.

35. The quantity of matter in any body is proportional to its magnitude and density conjointly.

When the magnitude is the same, the quantity of matter is as the density (*def. 3*), and when the density is the same, the quantity of matter is evidently as the magnitude. Therefore when neither is the same, the quantity of matter is as the magnitude and density conjointly ; or  $q = b d$  ; putting  $q$  for quantity of matter,  $b$  for bulk, and  $d$  for density.

#### PROP. 2.

36. In all uniform motions, the space is as the velocity and time conjointly ; or, as the number expressing the velocity multiplied by the number expressing the time.

For, if a body describe 10 feet in any unit of time, as a second, with a uniform velocity, it will describe  $10 \times 5$ , or 50 feet, in five seconds, with the same velocity unvaried.

If  $v$  = the velocity estimated by the space through which the body moves in a certain unit of time, as a second,  $t$  = the whole time or number of units, and  $s$  = the whole space or line passed over ; then  $s = t v$ .

**COR. 1.** The time is as the space directly and the velocity inversely; or, as the space divided by the velocity.

For, if a body describe 50 feet with the velocity of 10 Feet in a second; the time will be  $\frac{50}{10}$  or 5 seconds. Therefore  $t = \frac{s}{v}$ .

**COR. 2.** The velocity is as the space directly and the time inversely; or, as the space divided by the time.

For, if a body describe 50 feet in 5 seconds; its velocity will be  $\frac{50}{5}$  or 10 feet in one second. Therefore  $v = \frac{s}{t}$ .

### PROP. 3.

**37.** The momentum is as the velocity and quantity of matter conjointly.

For, the motion of the whole is the sum of the motions of all the parts; and therefore, in a body double in quantity, with equal velocity, the momentum is double; with twice the velocity, it is quadruple. Therefore, if  $v$  denote the velocity of a moving body,  $q$  its quantity of matter, and  $m$  its momentum; then  $m = q v$ .

**COR. 1.** Hence, if  $V$  denote the velocity of another body,  $Q$  its quantity of matter, and  $M$  its momentum; then, .....  
 $M : m :: Q V : q v$ , and  $m = q v$  generally.

**COR. 2.** The momenta are proportional to the velocities when the quantities of matter are the same. For when  $Q = q$ , .....  
 $M : m :: V : v$ ; and  $M = v$  generally.

**COR. 3.** The momenta are proportional to the quantities of matter when the velocities are the same. For when  $V = v$ , .....  
 $M : m :: Q : q$ ; and  $m = q$  generally.

**COR. 4.** The momentum and quantity of matter being given, the velocity may be found; also, the momentum and velocity being given, the quantity of matter may be found. For since  $m = q v$ ,  $v = \frac{m}{q}$ , and  $q = \frac{m}{v}$ .

### PROP. 4.

**38.** All impulses are measured by the quantity of motion they can produce or destroy.

If one impulse generate or destroy a certain quantity of motion, and another impulse generate or destroy twice or thrice that quantity, the latter impulse is double or triple of the former. Therefore, if  $f$  denote the impulse,  $f = m$ .





COR. Hence,  $f \doteq v$ , when  $q$  is the the same ; and  $f \doteq q$  when  $v$  is the same.

## SECTION 2.

### I. OF THE COMPOSITION OF SIMILAR RECTILINEAR MOTIONS.

39. *Axiom.* A body or point cannot move several ways at once in absolute space, all the parts of which are supposed to retain the same invariable position in universal space.

40. But, a point or body may describe two or more relative motions at the same time.

A man may slide on a sheet of ice from east to west, while the ice glides down the stream from north to south : the line of his motion in relation to the ice will lie east and west, and that in relation to the banks will lie north and south.

41. This combination of different relative motions is called the composition of motion ; because, in every point of the motion really pursued, the several relative motions are to be found.

In such cases, the motion is compared with at least two spaces. Any motion, produced even by a combination of several forces, must, when only one space or surface is assumed in the comparison, be contemplated as a simple motion.

#### PROP. 5.

42. Two uniform motions in the sides of a parallelogram compose a uniform motion in the diagonal.

*Fig. 2.* Let the point A describe AB uniformly in a given time, while the line AB is carried uniformly along AC in the same time, and always parallel to its former position AB. The point A, by the combination of these motions, will describe AD the diagonal of the parallelogram ABCD uniformly in the same time.

The velocities in the lines AB and AC are proportional to AB and AC, because these lines are described in the same time. Consequently, when the moving point has arrived at E half way

in the moving line  $AB$ , this line will be in the situation  $GH$  half way between  $AB$  and  $CD$ , and the point  $E$  in the place  $e$  half way in  $GH$ . Draw  $EeL$  parallel to  $A$ ; then since  $AE:AB::AG:AC$ , and the angle at  $A$  common, the parallelograms  $EG$  and  $BC$  are similar. Hence (Euclid 26, 6,) they are about the same diagonal  $AD$ , and the point  $e$  is in that diagonal.

In like manner it may be shewn, that, when the moving point has described any other part of  $AB$ , as  $AF$ , the moving line will have described a proportional part of  $AC$ , as  $AI$ , and be situated as  $IK$ , and the point  $A$  always in  $AB$  will now be at  $f$ . When the moving point has arrived at  $B$ , the line  $AB$  will be in the situation  $CD$ , and the moving point still in the diagonal will be at  $D$ . Hence, wherever the point  $A$  may be found in the moving line, it will still be in the diagonal  $AD$ .

And the motion in  $AD$  is uniform; for the motion in  $AB$  is uniform; and the parts of  $AB$  and  $AD$  described in the same time, are always in the same proportion to each other.

43. The velocity in the diagonal is to the velocity in either of the sides as the diagonal is to that side.

This is evident, since  $AD$ ,  $AB$ ,  $AC$ , are all uniformly described in the same time.

#### PROP. 6.

44. Two similar, varying motions, in the sides of a parallelogram, compose a similar motion in the diagonal.

*Fig. 3.* Let  $AB$ ,  $AC$ , be the spaces described by two varying motions in a given time; and let  $Aeig$ ,  $Afkh$ , &c., be parallelograms about the same diagonal  $AD$ . This diagonal will be described in the same time by a similar motion. Suppose each of the distances  $Ae$ ,  $ef$ , &c., to be described with a uniform motion, but let the velocity in  $Ae$ , be greater or less than the velocity in  $ef$ , and the same of the distances  $Ag$ ,  $gh$ , &c.; and suppose  $Ae$  and  $Ag$  to be described in the same time, and likewise  $ef$  and  $gh$ .

Then, if the velocities in the corresponding parts of the lines  $AB$ ,  $AC$ , are in a constant ratio, viz., . . . if velocity in  $Ae$ : velocity in  $Ag::$  velocity in  $ef$ : velocity in  $gh$ , . . . the motions in  $Ae$  and  $Ag$  (by prop. 5,) compose the motion  $Ai$ , and those in  $in$  and  $im$  (equal and parallel to  $ef$ ,  $gh$ ), compose the motion  $ik$ . Hence, the combination of the different motions in  $Af$ , with the different motions in  $Ah$ , compose the different motions in  $Ak$ ; and all the motions in  $AB$ , combined with those in  $AC$ , compose all the motions in  $AD$ .

Now, if the points  $A$ ,  $e$ ,  $f$ , &c., similar to  $A$ ,  $g$ ,  $h$ , &c. be taken near





to each other without limits, the motions in  $AB$  and  $AC$  will vary continually, and the diagonals of the small parallelograms will be infinitely short; yet it is clear, that while the cotemporaneous velocities are in a constant ratio, the motions in  $AB$ ,  $AC$ , will still compose a motion in the diagonal  $AD$ .

Again, because the spaces  $Ai, ik, \&c., = Ae, ef, \&c., = Ag, gh, \&c.$ , and the velocities are as the spaces described in the same time; the velocities in  $Ai, ik, \&c., =$  the velocities in  $Ae, ef, \&c., =$  the velocities in  $Ag, gh, \&c.$  And when the spaces are diminished, and their number increased without limits, the same ratio must exist. Therefore, the varying motion in the diagonal is similar to the varying motions in the sides.

## II. OF THE COMPOSITION OF SIMILAR FORCES.

45. The forces, which acting singly would produce the motions in the sides of a parallelogram, are called simple, elementary, or constituent forces; and by Monge, composants.

46. The resultant of several forces, is the effective force which results from their joint action.

47. The single force, which, acting in the direction of the line described by the joint agency of several forces, would cause that line to be described in the same time, is properly called the equivalent. Or, the single force which, being substituted for several forces, would produce the same effect, is their equivalent.

48. When either one or both of the forces which conspire to produce any motion are of the nature of pressures, the motion is strictly compound; and the resultant is frequently called the compound force.

Thus when a pendulous body descends through an arc of a circle, it is constantly affected by two forces, gravity and the string, which act in combination during every moment of the descent. Projectiles and revolving bodies are likewise examples of motions in which the combination is indicated in every part of the path described.

49. When several forces act in the same direction, the resultant is equal to their sum, and has the same direction; for the full effect of each is produced, and in one direction.

50. When two equal forces act in opposite directions, the resultant = 0, and no motion is produced. The motion which the one would produce by its single action, is prevented by the equal and opposite action of the other. And conversely, when no motion arises from the action of two opposite forces they are equal.

51. When two unequal forces act in opposite directions, the resultant is equal to their difference, and is in the direction of the greater.

The greater force may be conceived to be composed of two forces having the same direction, one of which is equal to the less, and the other to their difference. The effect of that, which is equal to the less, is solely to prevent the moving action of the less, and the effect of the other is a motion in the direction of the greater of which it is a part.

#### PROP. 7.

52. When several impulses in different directions are simultaneously applied to the same point, this point either remains at rest, or it moves in a single direction in absolute space, describing a right line.

If motion is generated, the initial motion (by 39) can have only one direction, and this direction (by 27) will continue unchanged.

#### PROP. 8.

53. When two impulses act at the same point, and have the directions and proportion of the adjoining sides of a parallelogram, the resultant will have the direction and proportion of the diagonal.

*Fig. 2.* That is, if  $AB, AC$  be the directions of two impulses and proportional to their intensities, so that . . . impulse in  $AB : AB ::$  impulse in  $AC : AC$ ; the diagonal  $AD$  will be the direction of the resultant. And force in  $AD : \text{force in } AB : \text{force in } AC :: AD : AB : AC$ .

For, the motion indicates the direction, and measures the intensity of the impulse in all cases. The motions in the sides of the parallelogram shew the direction of the elementary forces to be





in the sides; and the motion in the diagonal, (prop. 5) composed of the motion in the sides, shews this to be the direction of the resultant. And as these motions measure the forces, it follows that force in  $AD$  : force in  $AB$  : force in  $AC$  ::  $AD$  :  $AB$  :  $AC$ .

#### PROP. 9.

54. Two similar pressures, considered as moving forces, having the directions and proportion of the adjoining sides of a parallelogram, generate a similar pressure which has the direction and proportion of the diagonal.

Because the *direction of the motion* indicates the *direction of the moving force*, it becomes evident (from prop. 6) that the resulting pressure has the *direction* of the diagonal of the parallelogram, of which the elementary pressures have the directions of the sides. That is  $AB, AC, AD$ , (fig. 3) being the directions of the motions, they are also the directions of the forces.

Also, because every change of motion is proportional to the changing force (by 30), and because similar motions in the sides of a parallelogram compose a similar motion in the diagonal, (by prop. 6); it follows, that similar forces acting in the sides generate a similar force in the diagonal.

Moreover, the ratio of similar forces is the same as the ratio of the motions they produce (by 24); therefore it is manifest (from the demonstration of prop. 6), that the constant ratio of the force in the diagonal to the force in either of the sides, is the ratio of the diagonal to that side.

#### PROP. 10.

55. If a force equal to the resultant of two similar pressures, which act at an angle on a moveable point or particle, be applied to the same point in the direction opposite to that of their resultant, motion will be prevented.

And when there is rest, each of the three forces is equal to the resultant of the other two and is opposite to it.

The first part is evident from article 50.

*Fig. 4.* To prove the second part, suppose  $AB, AC$ , to be the directions and proportion of two pressures acting at the point  $A$ ; and suppose  $Ad$  to be a force equal and opposite to  $AD$  their resultant. Complete the parallelograms  $ABDC, ABcd$ . Then  $AC$  will be equal and opposite to  $Ac$  the resultant of  $AB$  and  $Ad$ .

For since  $Bc = Ad = AD$ , and  $Bc$  is parallel to  $dAD$  which is by hypothesis a right line,  $ADBc$  is a parallelogram. Therefore  $Ac (= BD) = AC$ .

And the angles  $cAB$ ,  $BAD$ ,  $DAc$  at the point  $A$ , are each equal to an angle in the triangle  $ADB$ ; that is,  $cAB = ABD$ ,  $CAD = ADB$ , and the remaining angle at the point  $A$  is the same as the remaining angle  $BAD$  of the triangle. Hence, the sum of the angles at  $A$  lying the same way = the sum of the angles in the triangle = two right angles. Therefore (Eu. 14, 1),  $C'A'c$  is a right line, and  $AC$  is opposite to  $A'c$ .

It can in like manner be shewn, that  $AB$  is equal and opposite to the resultant of  $AC$  and  $Ad$ .

*NOTE. Perhaps the following, in which the conditions of the forces alone are considered, may be to students a more useful method of demonstrating the proposition: and is applied in determining the truth of the next.*

If a point be at rest under the action of three pressures; let any one of them be withdrawn, and a motion will commence which is the effect of the resultant of the other two. Restore that pressure, and the motion ceases. Therefore (50) each of the three forces is equal to the resultant of the other two.

And each force is opposite to the resultant of the other two. For if not, one of the forces acts at an angle with the resultant of the other two, a new resultant is generated by that force and that resultant, and the new resultant must produce motion; which is contrary to the hypothesis. Therefore, each pressure is opposite to the resultant of the other two.

*This is the principle of the equilibrium of three pressures, whose directions concur in one point, at any angles in the same plane. The action of each pressure makes the point immoveable with respect to the action of the other two; which constitutes the equilibrium.*

#### PROP. 11.

56. If two pressures, acting on an immoveable point, have the directions and proportion of the sides of a parallelogram, the resultant will have the direction and proportion of the diagonal.

*Fig. 4. Let  $AB$ ,  $AC$ , represent the directions and intensities of two pressures which act on the point  $A$ , rendered immoveable either by one or several forces. If  $A$  be rendered immoveable by a single force  $Ad$ , this force is (prop. 10) equal and opposite the resultant  $AD$  of the two pressures; if by several forces, their single equivalent will produce the same effect if applied in*





a direction opposite to  $AD$ . Therefore, whether  $Ad$  be a single force or the resultant of several, it is equal and opposite to  $AD$ , the resultant of  $AB$  and  $AC$ , (by prop. 10.)

Now when  $Ad$  is withdrawn, the resultant of  $AB$  and  $AC$  is (by prop. 9) in the direction and proportion of  $AD$ , the diagonal in which the point  $A$  now moves. When  $Ad$  is restored, the point  $A$  becomes immoveable under the pressures  $AB$ ,  $AC$ .

But  $Ad$  being just  $= AD$ , the point  $A$  could not be at rest, if the *direction* of  $AD$  changed, when  $Ad$  is opposed to it, (by the demonstration of prop. 10.) Neither could  $A$  be at rest, if the quantity of  $AD$  changed when  $Ad$  is opposed to it, (51.) Hence, if two pressures, &c.

**57. The resultant of forces which act at an angle is always less than their sum.**

For, the diagonal of a parallelogram is less than both the sides. That is,  $AD$  is less than  $AB + AC$ . Hence, when forces act at an angle, there is always a loss of effect; and the greater the angle at which they act the loss of effect will be the greater, as the resultant will be the less.

**58. The directions of the elementary and resulting forces are in one plane.** For, the sides and diagonal of a parallelogram are in one plane.

**59. The sides of any triangle drawn parallel to the lines  $AB$ ,  $BD$ ,  $AD$  (Figures 2, 3, 4, 5,) which are proportional to the elementary and resulting forces, are also proportional to those forces.**

For, by the properties of parallel lines, two lines, drawn parallel to two other lines which are inclined at any angle, will be equally inclined: that is, they will make an angle equal to that contained by the others. Therefore, a triangle so constructed will be similar to either of the triangles  $ABD$ ,  $ACD$ ; and consequently its sides will be proportional to the forces.

**60. The sides of any triangle drawn perpendicular to the sides of  $ABD$  (or  $ACD$ ), or at any *equal* inclination to the sides of this triangle, are also proportional to the forces.**

For, two lines, perpendicular to two others inclined at any angle, must evidently have themselves the same inclination as those others.

Also, two lines equally inclined to two others must be themselves inclined at an angle equal to the angle made by the inclination of those others. Therefore, (as in 59.)

61. Each force is proportional to the sine of the opposite angle of this triangle. For, the sides of any triangle are proportional to the sines of the opposite angles.

62. Each force is proportional to the sine of the angle contained by the real directions of the other two.

*Fig. 2 to 5.* For  $AD : AB :: \text{sine of } ABD : \text{sine } ADB$ . Now the sine of  $ABD$  is the same as the sine of  $BAC$ , contained between the directions of  $AB$  and  $AC$ ; and the sine of  $ADB$  is the same as the sine of  $CAD$ ; also  $AB : AC$  or  $BD :: \text{sine of } ADB$  (or  $CAD$ ) : sine of  $BAD$ .

63. When two forces  $a$  and  $b$  act at the same point making any angle  $m$ , the analytical expression of the resultant will be  $\sqrt{a^2 + b^2 + 2ab \cos m}$ .

*Fig. 6.* For  $AD^2 = AB^2 + BD^2 \pm AB \times BP$   
 $= AB^2 + BD^2 + 2AB \times BD \times \cos m$ .

**PROP. 12.**

64. When two forces act at right angles, the resultant is equal to the square root of the sum of the squares of the elementary forces.

*Fig. 5.* For (by Euclid 47, 1.)  $AD^2 = AB^2 + BD^2$  therefore  $AD = \sqrt{AB^2 + BD^2}$  or  $= \sqrt{AB^2 + AC^2}$ .

This proposition may be thus expressed, calling the elementary forces  $a$  and  $b$ , and the resultant  $d$ ; ....  $d = \sqrt{a^2 + b^2}$ .

**PROP. 13.**

65. The elementary forces are to each other inversely as the perpendiculars on their directions drawn from the same point in the resultant.

*Fig. 6.* Let  $AB, AC$  be the elementary forces and  $AD$  their resultant; complete the parallelogram and draw  $DG, DP$  from the point  $D$  in the diagonal, perpendicular to the directions of  $AC$  and  $AB$ .





$\triangle ACD = \triangle ABD$ ; hence  $AC \times DG = AB \times DP$ , for by the Elements, triangles are to each other as the rectangles of their bases and altitudes.

Therefore  $AC : AB :: DP : DG$ .

If the perpendiculars are dropped from any other point  $d$ , then by similar triangles  $DP : DG :: dp : dg$ .

Hence  $AC : AB :: dp : dg$ .

#### PROP. 14.

66. If perpendiculars drawn from the same point to the directions of the elementary forces be inversely as those forces, that point will be in the direction of the resultant.

*Fig. 6.* Let  $AB, AC$ , be the elementary forces, and  $dg, dp$  perpendiculars from the point  $d$ . Then will  $d$  be in the diagonal. Complete the parallelogram  $ACDB$ , and draw  $dc, db$ , parallel to  $AB, AC$ .

Then because  $\angle gdb$  and  $\angle pdc$  are right angles and  $\angle cdb$  is common;  $gdc = pdb$ ; and the angles at  $g$  and  $p$  being right angles,  $\triangle gdc$  and  $\triangle pdb$  are similar.

Therefore  $dg : dp :: dc : db :: Ab : Ac$ . But by hypothesis  $dg : dp :: AB : AC$ .

Therefore  $Ab : Ac :: AB : AC$ . Therefore the parallelograms  $Abdc, ABDC$ , are similar, and (by Euclid 26, 6,) about the same diagonal. Consequently the point  $d$  is in the diagonal of  $ABDC$ ; that is, in  $AD$ .(w)

#### PROP. 15.

67. If two parallel forces be applied to different points in a right line and act the same way, the resultant is equal to their sum.

For, such forces cannot, like those inclined at an angle, diminish each other's effect; and as the full effect of each is produced, their resultant is equal to their sum.

COR. Since forces, which act at an indefinitely small angle, may be considered as parallel, their resultant is equal to their sum when they act the same way.

#### PROP. 16.

68. If two equal parallel forces be applied to the extremities of a right line, and both act the same way, the direction of the resultant will bisect the right line, and be parallel to the forces.

*The demonstrations marked (w) are by the author's son, William Barton Rogers.*

**Fig. 7.** Let  $AB$  be the right line, and  $P$  and  $Q$  equal parallel forces, applied to its extremities, and acting the same way. Bisect  $AB$  in  $C$ , and draw  $DE$  through  $C$  perpendicular to  $P$  and  $Q$ .

The forces  $P$  and  $Q$  may be considered as meeting in a point  $R$  infinitely distant from  $AB$ , or as being inclined at an infinitely small angle.

Since the angles at  $D$  and  $E$  are right angles, and the vertical angles at  $C$  are equal, (the sides  $CA$ ,  $CB$  being equal by hypothesis) the sides  $CD$ ,  $CE$  are equal. Because the forces  $P$  and  $Q$  are equal, and  $CD = CE$ , the point  $C$  is, (by prop. 14,) in the direction of the resultant.

Again, since every point in  $CR$  is equidistant from  $AR$  and  $BR$  in any inclination of these lines, it must still be so when the inclination vanishes.

Hence,  $CR$  is parallel to  $AR$  and  $BR$ ; that is,  $R$  the resultant is parallel to  $P$  and  $Q$  when these forces are parallel.

**NOTE.** If  $AB$  be a moveable inflexible line, and a force  $S$  equal to the resultant of  $P$  and  $Q$  be applied at  $C$ , but acting the contrary way, it will (50) prevent the motion which would arise from the action of  $P$  and  $Q$ ; and the line  $AB$  will remain at rest. *This is the principle of the equilibrium of the balance of equal arms.*

#### PROP. 17.

69. If two unequal parallel forces be applied to the extremities of a right line and act the same way, the resultant will divide that line in the inverse ratio of the forces, and be also parallel.

**Fig. 8.** Let the parallel forces  $P$ ,  $Q$ , be applied to the extremities  $A$ ,  $B$ , of the line  $AB$ , and act the same way. Divide  $AB$  in  $C$  in the inverse ratio of  $P$  and  $Q$ ; that is, make  $AC : BC :: Q : P$ , then  $C$  will be a point in the direction of the resultant. Through  $C$  draw  $DE$  perpendicular to  $P$  and  $Q$ , and suppose the directions of  $P$  and  $Q$  to be inclined at an infinitely small angle.

Then by similar triangles,  $DC : EC :: AC : BC$ ; but  $AC : BC :: Q : P$ ; therefore  $DC : EC :: Q : P$ . Hence (by Prop. 14) the point  $C$  is in the direction of the resultant.

Again, since every point in  $R$  the resultant has the same proportional distance from  $P$  and  $Q$  in any inclination of these lines, it must still be so when the inclination vanishes. Hence  $R$  is parallel to  $P$  and  $Q$ .

70. If the line  $AB$  be moveable and inflexible, the resultant  $R$  acting the same way as  $P$ ,  $Q$ , will move the line just





as these forces would move it. That is, a force  $= P+Q$  applied to the point C and acting alone, will produce the same motion, which P and Q applied at the extremities would produce. Otherwise, the point C cannot be the situation of the resultant.

71. Hence, if a force equal to the sum of the elementary forces be applied to the place of the resultant and act the contrary way, it will prevent the motion which the elementary forces would produce.

That is, if  $R$  be opposed to  $R$  or to  $P$  and  $Q$  in the point C, the line AB will be at rest. For (50) the resultant of  $R$  and  $R = O$ , and therefore the resultant of  $R$ , and  $P+Q = O$ .

NOTE. This is the principle of the equilibrium of the lever of the first order.

#### PROP. 18.

72. If a force  $R$ , equal to the resultant  $R$  of the parallel forces  $P$  and  $Q$ , act at the place of the resultant, and in the opposite direction; then either of the three forces  $R$ ,  $P$ , and  $Q$ , is equal to the resultant of the other two.

Fig. 9. That is, let  $R (= R)$  be applied at the point C, parallel to  $P$  and  $Q$ , and acting in the contrary direction.

$R = R$  = the resultant of  $P$  and  $Q$ .

$P = R - Q$  = the resultant of  $R$  and  $Q$  (by 51.)

$Q = R - P$  = the resultant of  $R$  and  $P$  (by 51.)

NOTE. It is evident that if, in the assigned directions, the three forces be applied to the same point C, there will be no motion; if  $R$ ,  $P$ , act at the same point C, then abstracting  $Q$ , there will be a motion in the direction of  $R$  the greater, equal to what  $Q$  would prevent if opposed also to  $R$ ; and if  $R$ ,  $Q$ , act at the same point, there will be a motion in the direction  $R$ ; equal to what  $P$  would prevent if opposed also to  $R$ .

If  $P$  and  $Q$  act upwards, one of them being a prop, while  $R$  is a pressure or weight; we have the principle of the equilibrium of the lever of the second order. Reversing the directions of all the forces, we have the principle of the equilibrium of the lever of the third order.

## PROP. 19.

73. When two unequal parallel forces are applied to different points in a right line, and act in contrary directions, the resultant is equal to their difference, and on the other side of the greater.

*Fig. 10.* Let  $P$  be the greater force and  $Q$  the less. Make  $AC : BC :: Q : R = P - Q$ . Then  $R$  the resultant  $= R = P - Q$  (by prop. 17 and 18;) and  $R$  is on the other side of  $P$  the greater acting the same way.

## PROP. 20.

74. When two unequal parallel forces are applied to different points in a right line, but in contrary directions, the distances of the resultant from those points are inversely as the forces.

*Fig. 10.* That is,  $AB : AC :: P : Q$ .

For  $R (= R) : Q :: BC : AC$ .

And (by comp.)  $R \times Q : Q :: BC \times AC = AB : AC$ .

But  $R \times Q = P$ .

Therefore,  $P : Q :: AB : AC$ .

Therefore, the point  $A$  where the resultant acts makes the distances from the points  $B$  and  $C$  inversely as the forces at those points.

## PROP. 21.

75. When three forces have the proportion and directions of the three adjoining linear edges of a parallelopiped, the resultant will have the proportion and direction of the diagonal.

*Fig. 11.* Let  $AB, AC, AD$  be three forces applied to the point  $A$  or whose directions meet at this point; the resultant will lie in the diagonal  $AE$ , of a parallelopiped  $H B$ , and be proportional to it.

Join  $AG, CE$ ; the resultant of  $AB, AD$  is the diagonal  $AG$  of the parallelogram  $DB$ , (by Prop. 7 and 8.) And since by the property of the parallelopiped (Euclid 10, 11,)  $CE$  the diagonal





of one surface is equal and parallel to A G the corresponding diagonal of the opposite surface, the figure A C E G is a parallelogram. Therefore A E is the resultant of A G and A C. Hence it is the resultant of A B, A D, A C.

**PROP. 22.**

76. When three forces act at right angles, the resultant will be equal to the square root of the sum of the squares of the forces.

*Fig. 11.* Let H B be the rectangular parallelopiped, of which the three forces are in the adjoining edges. Then (by Euclid 47, 1,)  $A G^2 = A B^2 + (B G^2 \text{ or } A D^2)$ ; and  $A E^2 = A G^2 + (E G^2 \text{ or } A C^2)$ .

Therefore  $A E^2 = A B^2 + A D^2 + A C^2$ .

Therefore  $A E = \sqrt{A B^2 + A D^2 + A C^2}$ .

This proposition may be thus expressed, calling the elementary forces  $a, b$  and  $c$ , and the resultant  $d \dots$ ;  $d = \sqrt{a^2 + b^2 + c^2}$ .

**III. HOW TO FIND THE RESULTANT OF SEVERAL FORCES.**

**PROP. 23.**

77. To find the resultant of any number of parallel forces acting the same way, compound two, then their resultant with a third, and so on.

*Fig. 12.* Let the forces O, P, Q, R, be applied to the points A, B, C, D which are invariably connected.

Take any two forces, as O, P, and find their resultant. This resultant will be equal O + P, (prop. 15.) and its direction parallel (prop. 17.)

Its place then is found by the following proportion;

$$O + P : P :: A B : A G.$$

$$\text{For } O : P :: B G : A G.$$

$$\text{And by comp. } O + P : P :: B G + A G = A B : A G.$$

Also  $O + P : O :: A B : B G$ ; now, in each of these two last proportions the three first terms are given.

Next, compound the resultant I thus found with another force Q; and we obtain the resultant  $K = I + Q = O + P + Q$ . The place of this resultant is in like manner found by this proportion,  $I + Q : Q :: G C : G H$ , or,  $O + P + Q : Q :: G C : G H$ .

Proceed in the same way to find  $R$  the resultant of  $H$ ;  $S$ , and its place  $M$ , and so for any number of forces whatever. The quantity of the last resultant will be equal to the sum of all the forces in the system; and the place of the last resultant will be their common centre; and is called the centre of parallel forces.

78. Hence, *first*, if the centre of parallel forces  $M$  is invariably connected with the points at which they act, a parallel force equal to the resultant  $R$  acting in the opposite direction will prevent motion.

*Second*. If some of the parallel forces act one way and the remainder the contrary way, having determined the particular resultant of each set, the difference of these resultants will be the general resultant; and a force equal and opposite to this general resultant will prevent motion.

*Third*. If the constituents, still parallel and the same in quantity, change their direction, the general resultant will remain the same in quantity, have the same place of application, and be parallel.

*Fourth*. If the points of application of the constituents are in the same plane, the centre of parallel forces will be in that plane also.

For, if the figure  $A B C D$  have all its sides in one plane, the right line  $G C$  and  $H D$  must lie in the same plane.

*Fifth*. If the points at which the parallel forces act are in the same right line, the centre of parallel forces, which is the place of the general resultant, is also in that right line.

#### PROP. 24.

79. To find the resultant of any number of inclined forces the directions of which meet at a point, and are either in the same plane or in different planes.

*Fig. 13.* Let  $O, P, Q, S$ , be the forces meeting at  $A$ . Take on their directions  $A B, A C, A D, A E$ , lines proportional to the forces respectively. Then compound two of them  $O, P$ , by completing the parallelogram; the diagonal is the particular resultant  $T$  of these two. Then compound  $T$  with another force  $Q$  in the same manner, and the diagonal is the resultant  $V$ , of  $T, Q$ , or of





**O, P, Q.** And lastly compound this resultant **V** with the force **S**; and the diagonal **AH** is the general resultant, **R**, of all the forces, **O, P, Q, S**.

80. To maintain the point to which several forces are applied at rest, find the quantity and direction of the general resultant, and apply a force equal to it in the opposite direction.

**PROP. 25.**

81. To find the resultant of several inclined forces, whose directions lie in one plane but do not meet in the same point.

*Fig. 14.* Let **O, P, Q** be forces acting at the points **A, B, C**, invariably connected; take distances **AD, BE, CF**, on their directions proportional to **O, P, Q**. Then, produce the directions of any two **O, P**, until they meet at **G**, and with the distances **GH = AD** and **GI = BE** make a parallelogram; **GK** the diagonal will be the direction and proportion of **S** the resultant of **O** and **P**.

Then, compound this resultant **S** with the force **Q** by taking distances  $= GK$  and **CF** on their directions from the point of concurrence **T**, and making the parallelogram; **R** the resultant of **S** and **Q**, or of **O, P, Q**, has the direction and proportion of **TL** the diagonal. In this manner, the resultant of any number of forces disposed according to the proposition may be found; for the last resultant will be the resultant of all the forces.

#### IV. RESOLUTION OF FORCES.

81. A motion or force is said to be resolved, estimated, or reduced, when it is conceived to be composed of two or more determinate motions or forces in other directions.

Thus, the mariner having sailed fourteen miles south west in one course sets down ten miles south and ten miles west nearly, and proceeds in the same way for every other course, always reducing the ship's actual path into two motions. He resolves every oblique course into meridional and equatorial directions, forming the two sides of a right angled triangle, of which the real motion is the hypotenuse. He performs this resolution expeditiously by means of a traverse table, in which the sides of right angled tri-

angles answering to every course and distance are calculated. At the end of twenty four hours, he adds all the southings into one sum and all the westings into another; he considers these sums as forming the sides of a right angled triangle, and again by his traverse table, finds what angle and what distance corresponds to these sides.

The miner proceeds in a similar manner, when he takes the plan of subterraneous operations by the help of his line, compass, and traverse table. Having ascertained the sum of all the northings or southings, the sum of all the eastings or westings, and also that of the several dips by the resolutions ready prepared in his traverse table, he learns how far the works proceed north or south, how far east or west, and how far to the dip. These directions correspond with the contiguous edges of a rectangular parallelopiped, and the whole oblique distance to the diagonal.

The mechanical philosopher, in the investigation of the complicated phenomena which often engage his attention, considers every motion, as compounded of two, and more frequently as compounded of three motions in convenient directions at right angles to one another. He also considers every force as resulting from the joint action of two or three forces at right angles. From this process, he obtains the two sides of a rectangular parallelogram or the three sides of a rectangular parallelopiped, and from these computes the position and magnitude of the diagonal. And in determining the relation of the forces concerned in the mechanic powers, in their simple or combined operation, it is always necessary to resolve some force, as the gravity of a heavy body, into two others at right angles. Hence, the whole theory of the action and efficacy of engines fundamentally depends on the principles of the composition and resolution of forces.

#### PROP. 26.

83. To resolve a given force (A B) into two others, which shall have given directions.

*Fig. 15.* Draw the lines A D, A C parallel to the given directions M N, M P, and complete the parallelogram C D. Then (prop. 8, 10, 11,) A B is equivalent to A D and A C, which have the given directions, and is resolved as required.

The preceding proposition is important, when we wish to find the efficacy of a force in a particular direction; or to reduce it to a particular direction. It is evident, that in order to do this, we have only to resolve the force into two, one of which is parallel to the given direction, the other at right angles to it. The latter can have no effect in the given direction, and therefore the other will express the whole effect.





## PROP. 27.

84. To resolve a given force ( $AB$ ) into three forces, whose directions meet in the same point, but do not lie in one plane.

*Fig. 16.* Let the required directions be  $EI$ ,  $EM$ ,  $EP$ . Now, any two of these are in the same plane, since they are right lines meeting in a point. Resolve  $AB$  into two forces, one of which  $AC$  is parallel to  $EI$ , the other  $AD$ , parallel to the plane  $EFGH$ . Then resolve  $AD$  into two forces  $Ap$  and  $Am$ , parallel to  $EP$  and  $EM$ .  $AC$ ,  $Ap$ ,  $Am$ , are the three required forces; for,  $AB$  is equivalent to  $AC$  and  $AD$ , and  $AD$  is equivalent to  $Ap$  and  $Am$ ; therefore  $AB$  is equivalent to  $AC$ ,  $Ap$ ,  $Am$ , and these three have the required directions.

85. When the three directions, to which any force is reduced, are at right angles and meet in one point, the given force has the direction of the diagonal of a rectangular parallelepiped, of which the three forces are the three contiguous linear edges. In this most useful method of reduction, the derived forces are commonly called the three coordinates.

Whether the forces meet obliquely, or at right angles,

Either of the derived forces :

is to the given force ::

as the linear edge of the parallelepiped in the direction of the derived force :

is to the diagonal.

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## SECTION 3.

### I. OF ATTRACTIVE AND REPULSIVE FORCES.

86. The operation of certain powers in nature, by which bodies and their particles if otherwise unrestrained, are caused to approach and adhere like magnets, is called attraction.

There are several different cases of attraction which are supposed to arise from the agency of different principles or powers present in bodies; as the attraction of gravitation, of cohesion, of magnetism, of electricity.

## PROP. 28.

87. A particle of matter tends towards a fixed centre of attraction, in the direction of the right line which joins its centre with the centre of attraction.

For, if the particle be free its motion from a state of rest will be in the direction of the moving force; and if restrained, its pressure or tendency to motion cannot be conceived to be in any other direction.

## PROP. 29.

88. The particles of matter are urged towards a fixed centre of attraction by forces, which are inversely as the squares of their distances from that centre.

Let  $EB, FD$  be concentric spheres, and their centre  $S$  the vertex of the similar cones  $ASB, CSD$ . By geometry, the sections or similar surfaces  $AB, CD$  concave towards  $S$ , are to each other as  $SA^2$  to  $SC^2$ , and all the right lines which can be imagined to proceed from  $S$  incident on  $CD$  must pass through  $AB$ .

Let a stratum of gravitating particles cover  $AB$ , and another stratum of similar and equal particles cover  $CD$ , each stratum being one particle in thickness. Then, since the attraction of  $S$  on particles in  $CD$  must (by prop. 28). act in right lines, passing through the area  $AB$ , all the particles in  $AB$  are attracted just as much as all the particles in  $CD$ .

Therefore, the attraction of a particle in  $AB$ : the attraction of a particle in  $CD$  :: number of particles in  $CD$ : number of particles in  $AB$  ::  $CD$ :  $AB$  ::  $SC^2$ :  $SA^2$ , that is, inversely as the square of the distance from  $S$ .

It is easy to see that this law must still obtain, if the centre of attraction and the particles attracted be in motion.

89. Terrestrial gravity is a continual force or pressure of the nature of an attraction.

For, we find that bodies at rest press incessantly on their supports, and when raised from the earth's surface and then abandoned, they descend again with a continually increasing velocity.

90. The force of gravity diminishes, as we recede to greater distances from the earth's centre.





This law, which might be deduced theoretically from the preceding principles, is proved very decidedly by observation.

We find, for instance, that the same mass will draw the rod of the spring steel-yard with a greater force near the poles than at the equator, the earth being an oblate spheroid compressed at the poles; and that the oscillations of a pendulum are performed in shorter times in a low situation than on the top of a mountain.

**91. Yet the force of gravity does not vary sensibly at *small distances* above or below the earth's surface. This is proved by our experiments on falling bodies.**

**92. The force of gravity at the earth's surface is of such intensity, as to cause a free body to fall through 16 feet and 1 inch in 1". It is also found that the velocities acquired by bodies falling freely, are as the times of descent;— that the spaces described are as the squares of the times; and therefore as the squares of the velocities; and that with the final velocity continued unvaried, the body will describe a double space, in the same time in which the velocity was acquired.**

**93. Bodies at the earth's surface (or equidistant from its centre) gravitate with forces directly proportional to their quantities of matter. For, the gravity of any body is the aggregate of the gravity of all the particles which compose it.**

**94. Bodies above the earth's surface (88, 93), gravitate with forces, which are directly as their masses, and inversely as the squares of their distances.**

**95. Repulsion is the operation of a power inherent in the particles of matter of every kind, by which two particles are prevented from co-existing in the same place, and by which homogeneous particles if unrestrained by other forces are caused to recede without limits.**

If there were no repulsive forces, every body would pass freely through every other, as light through glass, and two integrant particle might have a common centre.

It appears probable from phenomena, that repulsion decreases from the mathematical centre of an ultimate particle by some law of continuity, but that at and near the centre it is sufficient to counteract any attractions which can be opposed to it by homogeneous particles. Consequently, all such particles, and also those

of most kinds of heterogeneous matter are incapable of ultimate compenetration. Yet when repulsive forces do not operate, two heterogeneous particles, as a particle of light and a particle of glass, may exist in the same place. If the minute particles of light can penetrate the ultimate atoms of diaphanous bodies, without destroying, or displacing them, it follows that each particle may be considered as a sphere of repulsion, with respect to particles of its own kind, and therefore extended without being impenetrable to those of another kind.

The existence of a repulsive force extending beyond the apparent surface of dense substances, is an evidence that a power of repulsion unconnected with any central atom of a defined magnitude, may be sufficient for producing all the phenomena of tangible matter, in giving impulse and in resisting pressure and penetration.

There is no reason to doubt, that even the ultimate atoms of the densest bodies, are readily permeable to the causes of attractions of every kind, which have therefore been denominated semi-material substances. Thus, a magnet operates on a needle, as rapidly and efficaciously through a plate of glass or of gold, as if only a vacuum of equal thickness intervened.

96. A body or particle tends to recede from a centre of repulsion in the direction of the right line, which joins it with that centre. And the force by which a body or particle is repelled from a centre of repulsive force, is inversely as the square of the distance from the centre. For, a force directed from any centre must act according to the same laws, as one directed towards that centre.

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## II. OF ACCELERATING FORCES AND VARYING RECTILINEAR MOTION.

97. The motion of a body produced by the agency of any attractive or repulsive force is continually increased.

Thus, a heavy body falling from an eminence acquires in the first instant of time an infinitely small velocity. But, gravitation acting continually, there is in every succeeding instant an addition made to this velocity. Hence in a time that is just sensible, a very manifest velocity is generated; and hence also, the greater





the height from which the body falls, there is the more danger of its destroying whatever opposes its progress.

98. The motion of a body, projected in a direction contrary to that of any attractive force which acts on it, is continually diminished.

For example, the continual action of gravity on a stone impelled in a vertical direction gradually diminishes and finally extinguishes the upward motion. In the former case, the motion is said to be accelerated and the attraction is called an accelerating force; in the latter case, the motion is said to be retarded and the attraction is called a retarding force.

99. Uniformly accelerating forces, are proportional to the increments of velocity, which they produce in equal increments of time.

For, that force must be considered as double, which in the same increment of time, produces a double increment of velocity. Therefore, the accelerating force  $a = \frac{v}{t}$  when  $t$  is the same.

100. Uniformly accelerating forces, are proportional to the spaces described in equal times from the beginning of the motion.

Thus, if one force cause a body to describe 10 feet in a second, and another force would cause it to describe 20 feet in a second, the latter must act with twice the intensity of the former. Therefore,  $a = s$  when  $t$  is the same.

101. Uniformly accelerating forces, are inversely as the times, in which a given change of velocity is produced.

It is evident, that if a force generates a certain velocity in a certain time as a second, and another generates the same velocity in half the time, that is in half a second, the latter must have double the intensity of the former. Therefore  $a = \frac{v}{t}$  when  $v$  is the same.

102. Accelerating forces, are directly as the increments of velocity, and inversely as the increments of time.

This more general theorem is only a combination of the two preceding propositions, (99, 101.)

Suppose  $A, a$ , to be two pressures or constant forces, the one producing a greater, the other a less acceleration in the same time. To compare these forces when the observed times of their action are unequal, let  $V$  and  $v$  be the increments of velocity and  $T$  and  $t$  the unequal portions of time in which they are generated.

Then  $A : a :: \frac{V}{T} : \frac{v}{t}$  and hence  $a = \frac{v}{t} \cdot \frac{T}{V}$ .

PROP. 30.

103. If equal impulses act on a particle of matter, at equal intervals of time, and in the same direction; the velocity at any period of the motion will be proportional to the number of intervals, or to the time that has elapsed.

Let  $a$  be the space described with a uniform motion in the first interval by the action of the first impulse, the second impulse will just double its velocity (by 2 law.) Hence, during the second interval it will pass over the space  $2a$ ; during the third interval  $3a$ . Now, if  $t$  denote the whole time or number of intervals, the moving point will during the  $t$ th interval describe the space  $t a$ . And at any period of the motion, the number of equal increments of velocity will be equal to the number of the intervals of time elapsed.

PROP. 31.

104. If the number of intervals be exceedingly great, the space described will be nearly proportional to the square of that number; that is, to the square of the time elapsed.

The whole space will be equal to the sum of the arithmetical progression  $a, 2a, 3a$ ; carried on as far as the number of intervals extend. So that  $S = \frac{a + t a}{2} \times t = \frac{t a + t^2 a}{2} = a \times \frac{t + t^2}{2}$ .

Since  $t$  the number of intervals is very great,  $t^2$  is very great in comparison with  $t$ , and  $S = \frac{a t^2}{2}$  nearly. Hence also the space will be nearly proportional to  $t^2$ , since  $a$  and  $2$  are constant.

PROP. 32.

105. The space which the particle would describe with the last acquired velocity in the time  $t$  would be nearly double the space passed over, if  $t$  is very great.





Since the motion in this case is uniform, the space  $S = t v$  putting  $v$  for the acquired velocity.

And  $t = \dot{v}$  (by prop. 30)  $\therefore t a \times v = t^2 a = \text{twice } \frac{a t^2}{2}$ .

**PROP. 33.**

106. When a motion is uniformly accelerated, the space described from the beginning, is as the square of the time elapsed, or as the square of the last acquired velocity.

In this case, instead of impulses repeated at equal intervals, there must be a force in continual action, and of invariable intensity. Then it is evident, that the velocities added at the end of equal intervals will be equal. Hence, if  $t$  denote the number of instants in any finite portion of time, and  $v$  the final velocity; it follows (by demonstration of prop. 30) that  $v = \dot{t}$ ; also since in this case  $t$  is infinitely great (by prop. 31)  $s = \dot{t}^2$  exactly.

But  $v = \dot{t}$ ; hence  $s = \dot{v}^2$ .

**PROP. 34.**

107. The space, described in any time from the beginning of an uniformly accelerated motion, is exactly one half of the space which would be described in the same time with the final velocity unvaried. This follows from the demonstration of prop. 32.

108. If the times from the beginning of the motion are taken as the natural series of numbers; the acquired velocities will be as the doubles of these numbers; the whole spaces as the squares of those numbers; the spaces for each time as the odd numbers of the natural series, and the constant difference of those spaces will be 2.

If the times be as	1, 2, 3, 4, 5, 6, &c.
The acquired velocities will be as	2, 4, 6, 8, 10, 12, &c.
The whole spaces as	1, 4, 9, 16, 25, 36, &c.
The space for each time as	1, 3, 5, 7, 9, 11, &c.
Their constant differences as	2, 2, 2, 2, 2, &c.

109. The relations of the spaces to the times or to the velocities in a motion uniformly accelerated, may be expressed by the abscissæ and ordinates of a parabola.

*Fig. 18.* Let  $AB$  be a parabola,  $AC$  its axis, and  $CB$  an ordinate; then  $AE$ ,  $AF$ ,  $AC$ , the abscissæ,  $= Ee^2$ ,  $Ff^2$ ,  $CB^2$ . Let  $ae$ ,  $af$ ,  $DB$  be parallel to the axis; if these lines represent the spaces described,  $Ee$ ,  $Ff$ ,  $CB$  will represent the times or the acquired velocities. For  $ae : af : DB :: Ee^2 : Ff^2 : CB^2$ .

110. It is very common to represent the relations of space, time and velocity, in uniformly accelerated motion, by the area and sides of a right angled triangle. And although the representation of linear space by an area does not appear to be natural, it is nevertheless a true one as respects the relations of the quantities of  $s$ ,  $t$ , and  $v$ .

*Fig. 19.* In the triangle  $ABC$ , let  $AB$  represent the time from the beginning of the motion, and  $BC$  the last acquired velocity. Draw any number of lines parallel to  $BC$ , as  $Ee$ ,  $Ff$ ,  $Gg$ , &c. By similar triangles  $AE : Ee :: AB : BC$ . Hence, if  $AE$  represents a part of the time  $AB$ ,  $Ee$  will represent its corresponding velocity; and so of the other lines  $Ff$ ,  $Gg$ , &c. Again, the triangle  $ABC$ , being equal to  $\frac{1}{2} AB \times BC$  that is  $= \frac{1}{2} t v$ , it is the representative of the space described in the time  $t$  which is represented by  $AB$ . And because  $AEe : ABC :: AE^2 : AB^2$ ,  $AEe$  will (from prop. 33) represent the space  $S$  described in the time represented by  $AE$ ; also,  $AFf$  will represent the space described in the time  $AF$ .

Also, the rectangle  $ABCD = 2ABC$  represents the space  $2s$  which would be described in a time  $= AB$  with the final velocity  $BC$ .

PROP. 35.

111. In similar motions compared with each other, the acquired velocities are as the spaces described in equal times from the beginning.

That is,  $V : v :: S : s$ .

*Fig. 20.* Let the two lines  $AB$ ,  $ab$ , be described by similar motions in the same time. In  $AB$  take any two points  $C$  and  $D$ , and in  $ab$  take the similar points  $c$  and  $d$ : then  $CD$ ,  $cd$ , are described in the same time. If the points  $D$ ,  $d$ , always similar,





continually approach  $C, c$ , the ultimate ratio of  $CD$  to  $cd$  will be the ratio of the velocities at  $C, c$ ; that is  $V : v :: CD : cd$ .

Now  $AD : AC :: ad : ac$ , (def. 21).

Therefore  $(AD - AC) = CD : (ad - ac) = cd :: AC : ac$ .  
Consequently  $V : v :: AC : ac$ . (w.)

PROP. 36.

112. In a motion continually varied, the velocities in the different points of the path are to each other in the ultimate ratio of the spaces described in equal times, those times being supposed to diminish continually.

For, when the equal times are diminished without limits, the velocities are uniform. Hence, just before the spaces vanish, their ratio is the ratio of the velocities.

For a full demonstration see Robison's Dynamics.

PROP. 37.

113. Uniformly accelerating forces compared with each other are as the velocities directly and as the times inversely, or as the velocities divided by the times.

For (102)  $A : a :: \frac{V}{T} : \frac{v}{t}$ . But when  $A$  and  $a$  are uniformly accelerating forces, the increments of time and velocity will have the same proportion as the whole time and velocity, therefore,  $\frac{V}{T} : \frac{v}{t} :: \frac{V}{T} : \frac{v}{t}$ , consequently  $A : a :: \frac{V}{T} : \frac{v}{t}$ .

Hence,  $a \doteq \frac{v}{t} \doteq v$  when  $t$  is the same,  $\doteq \frac{1}{t}$  when  $v$  is the same.

114. We have so far considered acceleration simply, that is, the continual augmentation of velocity; and we have regarded the accelerating force as acting on equal particles or masses. But, since the *quantity of motion* measures the *magnitude of the force*, a difference in mass will require a difference in the absolute quantity of force to produce the same acceleration.

Thus, if B contain 2,000 particles, and  $b$  contain 1000 equal particles, and both are equally accelerated, then  $F = 2f$ , or the force which moves, B is twice as great as the force which moves  $b$ .

PROP. 38.

115. The absolute quantity of a moving force, is as the mass into the velocity directly; and as the time inversely, when the acceleration is the same.

That is,  $f \doteq q \frac{v}{t}$ ,  $\doteq \frac{q v}{t}$ ,  $\doteq \frac{m}{t}$  when  $m$  is the momentum.

For the equal particles of two bodies, B and  $b$ , are each moved by a force  $= \frac{v}{t}$ . Let Q be the number of particles in B, and  $q$  the number of equal particles in  $b$ . Then the aggregate of all the forces which move the particles of B  $= Q \frac{v}{t}$ , and of those which move the particles of  $b = q \frac{v}{t}$ .

Call these aggregate forces F and  $f$ , and  $F : f :: Q \frac{v}{t} : q \frac{v}{t}$ .

Hence,  $f \doteq q \frac{v}{t}$  or  $\doteq \frac{m}{t}$ .

PROP. 39.

116. The momentum generated by a constant moving force, (that is, by the aggregate force  $f$ ,) is in the compound ratio of the force and time.

The momentum generated in an instant, is as the force; for, a double momentum, generated in an instant, indicates a double force, and  $n$  times the momentum generated in an instant, indicates  $n$  times the force. (30.)

Again, as the moving force is constant, equal momenta will be generated in equal increments of time, and the whole momentum will be as the whole time elapsed.

Therefore  $m \doteq f t$ .

Or, the proposition may be deduced from the preceding one.

Since  $f \doteq \frac{m}{t}$ ,  $m \doteq f t$ .





NOTE. The accelerating force  $\frac{v}{t}$  is measured by the acceleration or velocity, and is not conceived to vary with the mass; thus, different masses are equally accelerated by terrestrial gravity. The aggregate, or moving force  $q \frac{v}{t}$  is measured by the momentum. Thus, one pound and one ounce are equally accelerated by gravitation. But the former acquires sixteen times the momentum of the latter, in the same time, and the moving force of the former, is therefore sixteen times as great as that of the latter.

PROP. 40.

117. The spaces which bodies describe by the action of constant moving forces, during any times, are in the compound ratio of the forces and squares of the times, directly, and the masses, inversely. That is,  $s \doteq \frac{f t^2}{q}$ .

For, (by prop. 39,)  $m$  or  $q v \doteq f t$ .

Therefore, . . . .  $v \doteq \frac{f t}{q}$ .

Therefore, . . . .  $t v \doteq \frac{f t^2}{q}$ .

But, . . . .  $t v \doteq s$ .

Therefore, . . . .  $s \doteq \frac{f t^2}{q}$ .

118. The laws of uniformly retarded motions, are the same as those of uniformly accelerated motions, only substituting decrements of velocity for increments. For, motion is generated and destroyed according to the same laws.

119. The time in which any velocity would be extinguished by a retarding force, is equal to the time in which the same force would generate this velocity in the body previously at rest.

Thus, if the final velocity of a falling body is equal to 32 feet in one second, the body has fallen through 16 feet in 1", and if it

be projected in a vertical direction with an initial velocity of 32 feet in 1" it will ascend just 16 feet in 1', and at that height and at the end of that time its upward motion will cease.

120. The times in which different initial velocities will be destroyed by the same retarding force, are as the initial velocities.

121. The distances to which the body will proceed before the entire extinction of its velocity, are as the squares of the initial velocities.

122. The distances are also as the squares of the times elapsed.

123. The distance to which a body will proceed before its motion is extinguished, is one half the space which it would describe in the same time, with the initial velocity continued uniform.

124. We are not acquainted with any force in nature which is, in mathematical strictness, a *uniformly* accelerating force.

The attraction of a magnet increases, as the iron approaches it; the repulsion of an electrified ball lessens as the repelled body recedes from it; the pressure of a spring diminishes as it unbends; and terrestrial gravitation is not of equal intensity at different distances from the earth's centre.

125. The force of gravity does not, however, differ sensibly in the intensity of its action at any distances to which our experiments extend. The actual decrease of the gravity of a body at the distance of a mile above the earth's surface does not, according to theory, exceed the 2000th part of its whole weight at the earth's surface. This force may, therefore, be considered as acting with equal intensity in every instant of the ascent or descent of the bodies which we subject to its influence; and all the rectilinear motions, which we observe it to produce, as uniformly accelerated.





## SECTION 4.

## I. OF THE COMPOSITION OF DISSIMILAR FORCES.

126. So far as we have considered the combination of transverse forces in the second section, the cases relate only to the action of similar forces, and the motion which arises is always rectilinear ; but a more interesting variety of motion, arises from the combined action of dissimilar forces ; as

*First.* From the joint action of an impulse and a pressure.

*Second.* From the combination of a constant pressure with one of variable intensity ; and

*Third.* From the joint action of two pressures whose intensities vary by different laws.

In all such combinations of forces, the motions are curvilinear ; and the character of the curve described by the moving body depends on the relations of the forces to one another, with respect to direction, intensity, constancy and variable-ness.

127. To illustrate the first case, that is the combination of an impulse and a pressure, or, of a uniform and an accelerated or retarded motion.

**Fig. 21.** Suppose a body at A to have received an impulse in the direction A B, causing it to move in that line with a uniform velocity, and when it has arrived at the point B, let an impulse P be applied in the direction P B of such intensity, that the body would by its single action describe B H, while with its original motion, it would describe B D ; then it will go through the diagonal B C, uniformly in the same time. If a succession of transverse impulses Q, R, S, T, be applied at finite intervals, all acting the same way, then there will be a succession of angular deflections always inclining the same way, and the motion in the intervals between the action of the impulses will be rectilinear and uniform, and always in the diagonals of parallelograms, whose sides are the directions of the previous motion and that which the new impulse would alone generate.

If we again suppose the intensity of these impulses to be exceedingly small, compared with the primitive force in A B, and that they are repeated not at finite intervals of time, but in immediate succession without any interval during a finite time, then

instead of angular deflections, there will be a continued flexure ; and instead of a polygonal path, the body will describe a curve. Now, an effect of this kind arises from the agency of an unintermitting attraction, which during the period of its action, produces a *continual* deflection from the momentary direction of the motion. The motion, at any point in the curve, is composed of a motion in the direction of the tangent at that point, and a motion in the direction of the pressure.

128. To illustrate the second case ; that is, the combination of a constant pressure with one of variable intensity.

Suppose a body to move in the direction H B, with a motion uniformly accelerated, and when it has arrived at A, let it be deflected in the direction A C, by an unintermitting force of such intensity that while A B would be described by the former motion, the transverse force would cause it to pass over A C. Then, if the transverse force were constant, the body would be at D, when, by the former motion it would have reached B, and at *d* when it would have reached *b*.

Now, suppose the transverse force to be variable and that its intensity diminishes during the whole time of its action, then the body will not be found at D in the diagonal A D, when by the motion in A B it would have arrived at B ; but it will be at some point F, less remote from B ; on the other hand, if its intensity augments, it will be found at some point G more remote from B.

Hence it is obvious, that if the transverse force *continually* decreases or increases, the deflections from A B, will never be in the diagonal, but always in some curve A F, bending towards A *b* in the first condition, and in a curve A G, bending from A *b* in the second condition.

129. To illustrate the third case ; that is, the combination of two pressures whose intensities vary by different laws.

First, suppose that A B, A C, are similar varying pressures ; then (prop. 9,) the motion will be rectilinear, and in the diagonal A D of the parallelogram A B C D. If now the intensity of A'B be supposed to increase continually, in such manner, that the ratio in similar points of space is no longer constant ; it is evident, that the body will suffer a continual deflection from A D towards A B ; and while, with the ratio constant, the body would have described A *d*, with the variable ratio it will describe some curve A F, bending towards A *b*.





## PART II.

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### DOCTRINE

OF THE

## FREE MOTION OF MASSES.

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Free motion is that which takes place in unresisting media ; the moving body being unconnected with any other. The existing condition of any free motion is affected only by attractions and repulsions acting at a distance.

### SECTION 1.

#### I. OF THE MOTION OF PROJECTILES.

130. When we observe a body in motion to bend its course continually from the right line, with which its momentary direction coincides, we unavoidably infer the agency of some incessant force, acting transversely with respect to the momentary direction of the motion. This transverse force, whether it be an attraction or repulsion, is, in reference to the effect it has on the path of the body, called a *deflective* force. The most important cases of deflected motion are those which are caused by terrestrial and planetary gravitation.

A body projected in any direction, not perpendicular to the horizon, is observed to be continually deflected from the line in which it was impelled. This effect is produced by the power of gravity, by which the body is continually retarded in its ascent, accelerated in its descent, and made to describe

a curve during the whole period of its flight. A body propelled by any temporary force, and afterwards left to proceed without farther incitement to motion, is called a projectile ; as an arrow, impelled by the elastic spring of the bow, or a metallic ball, moved by the expansive force of gun-powder.

In treating of the motion of projectiles, it is convenient to consider them, in the first place, as moving in an unresisting medium, and afterwards to estimate the effect of atmospheric resistance. Also, gravity is regarded as a constant force acting on the projectile in every point of its course perpendicularly to the plane of the horizon.

**PROP. 41.**

131. When a body is projected either horizontally or obliquely, the horizontal velocity remains invariably the same, notwithstanding the deflections which may be produced by gravity.

*Fig. 23 and 28.* For the velocity of projection is uniform. And if the direction be oblique, as  $A B$ , when this velocity is resolved into two,  $A E$  and  $A C = B E$ , the one horizontal, the other perpendicular, it is plain that the action of gravity can affect the latter only. If the direction be horizontal, as  $V T$ , gravity adds a perpendicular velocity  $T P$ , producing a compound motion ; but, it does not change the quantity of the horizontal velocity.

**PROP. 42.**

132. The place of a projectile in its path is always in the vertical line, passing through that point in the line of projection, which terminates the space it would have described in the time elapsed, had it not been deflected.

*Fig. 23.* Let  $A B$  be the direction of the projectile force, and  $A D C$  the path of the body. Since the horizontal velocity is constant, it follows, that when the body by its uniform and entire motion in  $A B$  would have arrived at  $B$ , it will be found at  $D$  vertically under  $D B$ .





## PROP. 43.

133. The velocity of a projectile in its path, during its ascent and descent, is always the same at equal altitudes.

*Fig. 23.* For, that part of the velocity of projection, which is horizontal, is constant; and at V the highest point of the path, the perpendicular part of that velocity is extinguished. Now, by the laws of uniformly accelerated and retarded motion, the time of destroying the perpendicular velocity, that is the time of ascent to V, is equal to the time of producing an equal perpendicular velocity in a contrary direction, that is, the time of descent from V to C. And because gravity is a uniform force, the decrements of velocity during the ascent to V, are equal to the increments after passing V, in equal portions of time.

## PROP. 44.

134. The path of a projectile moving in free space is a parabola.

*Fig. 24.* Let a body be projected from A in the direction of the line AD, and let A E G be the path described. Now, in uniform motions the spaces are as the times, and in uniformly accelerated motions the spaces are as the squares of the times. Therefore, the deflections caused by gravity, are as the squares of the spaces which would be described by the force of projection alone.

Since A B, A C, A D, = the times, and B E, C F, D G = the squares of the times; .....  $B E : C F : D G :: A B^2 : A C^2 : A D^2$ ; which is a property of the parabola, A D being a tangent at the point A, and B E, C F, D G, lines parallel to the axis.

## PROP. 45.

135. The velocity of a projectile in any point of its path is as the secant of the angle, which its direction in the tangent at that point makes with the horizon.

*Fig. 25.* Let A B be taken in the direction of the tangent, which is the momentary direction of the motion in the curve at the point A; and resolving the velocity at this point, represented by A B, into A C the horizontal, and B C the perpendicular velocity; then  $A C : A B :: R : \text{Sec. } B A C$ .

But A C and R are constant quantities; therefore  $A B = \frac{A C}{\text{Sec. } B A C}$ .

## PROP. 46.

136. The perpendicular velocity acquired or destroyed at any point is to the initial velocity of projection, as twice the deflection at that point to the space which would be described with the projectile velocity unchanged; or, as twice the absciss to the ordinate at that point of the curve.

*Fig. 26.* That is, if  $v$  be the perpendicular velocity acquired or destroyed at  $C$ , and  $V$  the initial velocity at  $A$ . Then  $v : V :: 2BC : AB :: 2AD : DC$ .

For, first, the time of describing  $AB$  and  $BC$  is the same; second, the acquired velocity at  $C$  would carry the body through  $2BC$  with a uniform motion in the same time; and third, in uniform motions the spaces are as the velocities.

Therefore,  $2BC : AB :: v : V$ .

And  $2AD : DC :: v : V$ .

## PROP. 47.

137. The velocity of a projectile at any point of its path is equal to that which would be generated by gravity, in descending freely through one fourth of the parameter of the diameter to that point in the parabola.

*Fig. 27.* Take any point  $A$  in the parabola  $EAC$ ; and let  $GA$  be the height from which a body must fall to acquire a velocity equal to that in the curve at  $A$  or in the tangent  $AB$ . Produce  $GA$  to  $D$ , making  $AD = AG$ ; also make  $AB = DG = 2AG$ , and complete the parallelogram  $ABCD$ .

Since  $AB = 2AG$ ; (by Dynamics)  $AB$  expresses the velocity at  $A$ . Because  $BC (= AD = GA)$  is the deflection which would be produced by gravity in the time of describing  $AB$ , it follows that  $C$  is a point in the path of the projectile. And because  $C$  is in the parabola,  $DC$  is the ordinate corresponding to the absciss  $DA$ . But  $AD (= \frac{1}{2} AB) = \frac{1}{2} DC = \frac{1}{2} EC$ . And (by conics) when any absciss  $AD$  is equal to  $\frac{1}{2} EC$ , the double ordinate, this line is equal to the parameter of the diameter  $AF$ .

Therefore, the velocity in the curve at any point  $A$  = the velocity generated in falling through  $(AD =)$  one fourth of the parameter to the diameter to that point.





## PROP. 48.

138. The velocity of a projectile in any point of its path, is equal to that which would be acquired by falling from the directrix to that point.

*Fig. 28.* Let PVP be a parabola, DD the directrix, F the focus and V the vertex. Then (by conic sections)  $DP = PF = \frac{1}{2}$  the parameter of the diameter to the point P; and the velocity acquired in falling through DP one fourth of the parameter = (by prop. 47.) the velocity of the body in the curve at P.

COR. As the velocity in V, the vertex of the curve, is horizontal,  $2DV$  measures the constant horizontal velocity.

## IK. OF CENTRAL FORCES AND OF MOTION IN ORBITS, INCLUDING THE FUNDAMENTAL PRINCIPLES OF PHYSICAL ASTRONOMY.

139. It is a familiar observation, that the greater the projectile velocity, the farther does the projectile range on the earth's surface. We can easily conceive, that the velocity of projection might be such, as would carry a body over a great part of the earth's circumference, or even entirely around it, before it struck the ground; and that by some additional force, the momentum might be such, as would cause it to return to the place from which it was projected, with undiminished velocity, supposing the atmosphere to give no resistance. In this case, it would proceed a second time from the same place with the original velocity, and of course it would again return with that velocity unimpaired. Hence, it would continue like the moon to revolve round the earth forever.

140. When a body in motion being continually deflected from its momentary direction in the tangent, still advances in the curve until it returns to the same point, and then repeats its motion in the curve, its path is called an *orbit*, and the line in which it moves, is said to return into itself, like a *circle* or an *ellipse*.

141. The motion from any point in an orbit continued round to the same point, is called a *revolution*, and the time in which a revolution is described, is called the *periodic* time.

142. The deflecting force which retains a body in its orbit, is called a *centripetal* force. In the great motions of the planetary system, the centripetal force is always an attraction directed to the centre of the system; and hence, the point to which this force in every part of the system is directed, is called the *centre of attraction*.

In magnetic and electric experiments, the deflecting force may be a repulsion as well as an attraction. In mechanical experiments, the deflecting force is commonly the cohesion of the parts of the instrument; as, when a revolving body is retained in a circular path by a string, or when bodies are put in circular motion by the whirling table.

143. The *projectile* force is strictly that, which primarily produces the uniform motion in the direction of the tangent. But, the momentum of a revolving body, by which according to the first law of motion, it would persevere in the direction of the tangent at any point of its orbit if not deflected, is also frequently called the projectile force at that point.

NOTE. It is the momentum in the direction of the tangent, or the tendency to persevere in that direction, which prevents revolving bodies from falling to the centre of attraction with an accelerated motion.

144. That effect of the momentum of a revolving body, which, opposing the action of the centripetal force, prevents the body from falling to the centre of attraction, and causes some particular orbit to be described, is called the *centrifugal* force.

145. The right line, which is conceived to join the centre of a revolving body, with the centre of attraction, is called the *radius vector*.

It is obvious, that the radius vector must describe triangular areas, while the revolving body describes corresponding arcs of its orbit; just as the string of a pendulous body in each oscil-





lation describes the sector of a circle, while the body passes through the arc of that sector.

146. The velocity of the radius vector, by which it generates an angle, is called the *angular velocity* of the revolving body.

When the motion of the radius vector is uniform, the angular velocity is as the magnitude of the angle generated in any unit of time. And when the motion of this line is variable, as is the case in all free motions in nature, the angular velocity of the body at any point of the curve, is estimated by the angle which is generated in an instant.

**PROP. 49.**

147. The centripetal and centrifugal forces are equal, and act in opposite directions.

*Fig. 29.* For, if a tangent touch the orbit, the actual deflection from it in an instant, is equal to the space which the centripetal force alone would cause the body to describe; and also, to the space which the centrifugal force has prevented it from passing through in the direction towards the centre in the same instant.

Let a body, revolving in the curve  $A C E$  about  $S$ , the centre of attraction, cease to be deflected by the centripetal force, and it will proceed in the tangent  $A T$ . Take  $A B$  indefinitely small, and draw  $B C$  parallel to  $A S$ . Then, when the body would have been at  $C$  by the concurrent action of the centripetal force, it will be at  $B$ , and  $B C$  is the distance to which the body would be removed by the single action of the centrifugal force. Complete the parallelogram  $B D$ ; and  $A D = B C$ , is the distance through which the body would descend towards  $S$  by the single action of the centripetal force, while  $A B$  would be described with the velocity in the tangent, or  $A C$  by the combined forces. Hence,  $2 B C = 2 A D$  measures both the centripetal and centrifugal force. They are, therefore, equal and opposite in direction. *Hence, the term central force is usually employed in announcing the laws of either, and the centre of attraction is called the centre of forces.*

148. Considering the momentum at any point as a projectile force in the direction of the tangent, if we resolve it into two others, the one in a direction opposite to the centri-

petal force, the other in the direction of the chord of an infinitely small arc of the orbit; the former becomes the centrifugal force, which is destroyed or countervailed by the centripetal force, and the latter is the effective moving force by which the body advances in the curve.

149. The centrifugal force can never be equal to the whole momentum of the revolving body, because the tangent can never have a direction opposite to that of the centripetal force, but always at an angle with it.

If the whole momentum of a revolving body had a direction opposite to that of the centripetal force, the body would either remain at rest, or descend to the centre, or recede from it in a right line; as is evident from the principles of the composition of opposite forces.

**PROP. 50.**

150. If a body revolve in a circular orbit, the centre of attraction coinciding with the centre of the circle, the momentum in every point of the orbit is the same, and is as the primary projectile force.

For, the initial momentum is as the primary projectile force, since it is the measure of that force, and in this case the centripetal force is always the same; hence, if the momentum became less at any point, the body would be more deflected and would deviate from the circular path towards the centre; and if the momentum became greater, it would deviate from the circular path by receding to a greater distance from the centre.

**PROP. 51.**

151. But when the centre of attraction does not coincide with the centre of circle, or when the orbit is elliptic, the momentum varies in different parts of the orbit.

*Fig. 30.* Suppose A B C to be a circular or elliptic orbit, and S the centre of attraction more remote from A, one extremity of the diameter or axis than from G the other extremity. The momentum of the body moving from A towards B, is less at B than at C; because the body approaches nearer to S, and therefore, has its motion accelerated. The momentum in the tangent





BE or in the tangent CE, may be considered as composed of two motions, the one equal to the momentum in AD, the other an acquired motion. The quantity of the former is constant, that of the latter variable. This variable part of the motion begins to exist in the nascent arc Aa, just as the perpendicular velocity of the descending projectile begins to exist as it proceeds from the highest point of its path. This variable part of the motion is also like the perpendicular velocity of the projectile continually accelerated, until it arrives at G, at which point it is nearest S. The body now begins to recede from S, and consequently the centripetal force becomes a retarding force. During the time of describing GHN, the other half of the orbit, the centripetal force continually diminishes the variable motion which it had produced; and at A it is altogether extinguished.

**PROP. 52.**

152. If a body revolve around a fixed centre of attraction, the areas, described by the radius vector, will be proportional to the times and lie wholly in one plane.

*Fig. 31.* Let us first suppose, that the body describes a polygon ABCFG, and that the sides AB, BC, &c., are described in equal times. Make BD = AB, and join CD and SD.

Had the motion in AB suffered no change in the point B, the body would have described BD in the next equal interval of time, after describing AB. But, an impulsive force having at that point acted in the direction BS, the diagonal BC of the parallelogram ED is described in that interval.

The triangle SDB = the triangle SAB, because they have equal bases and the same altitude; and the triangle SBC = the triangle SDB, because they have the same base SB and are between the same parallels SB and CD. Therefore, the triangle SBC (= the triangle SDB) = the triangle SAB. And they are described in equal times, since the time in BC = time in BD = time in AB.

And if we suppose all the deflections to be produced by impulses directed towards S and acting at equal intervals; it can in like manner be shewn that the successive deflected lines of motion CF, FG, &c., are the bases of triangles, each of which is equal to SAB. Hence, a double area will be described in a double time, and the area described in any given time will be proportional to the time.

Also the areas are all in one plane. For, the triangles SAB and SDB are in one plane, because AD is a right line; and the triangles SBC and SDB are in one plane, because they are be-

tween the same parallels; therefore, the triangles  $SBC$  and  $SAB$  are likewise in one plane.

Finally, suppose the equal intervals between the impulses to be infinitely small, or let the deflections be produced by a continual attraction towards  $S$ ; then, the triangles described in successive instants will be infinitely small. But, since equal finite triangles, however small, are described in equal minute intervals of time, and since the aggregate area of any number of these is proportional to the time of description, and also since they still are necessarily in the same plane, it is clear that the same relations must hold, when the number of the triangles is increased and their bases diminished without limits so as to compose a continued curve.

**PROP. 53.**

**153.** If the areas, described by the radius vector around any point, are all in one plane and proportional to the times, that point is the centre of attraction.

*Fig. 31.* For, every body that moves in a curve is (by 1st law) deflected from its rectilinear direction by some transverse force acting incessantly. And that force which causes the indefinitely small triangles of equal area to be described in equal moments of time, around the fixed point  $S$  (by prop. 40. 1 Euclid, and by 2nd law) acts at  $B$  in the direction of a line parallel to  $DC$ , that is, in the direction of the line  $BS$ ; and for the same reason it acts at  $C$  in the direction of the line parallel to  $FK$ , that is in the line  $CS$ , and so on. Therefore, the transverse force acts always in the direction of lines proceeding from the body in every part of the curve to the fixed point  $S$ .

**PROP. 54.**

**154.** The velocities at different points of the curve, are inversely as the perpendiculars drawn from the centre of attraction to the tangents at those points.

*Fig. 31.* For, the velocities in the lines  $AB, BC, CF, FG$ , &c. are as those lines; and since  $AB, BC, CF$ , &c., are the bases of equal triangles, they are inversely as the altitudes of the triangles, that is, as the perpendiculars let fall from the point  $S$  on their directions, (by Euclid 15, 6.) Consequently, the velocities are inversely as those perpendiculars, and ultimately (the bases being diminished, and their number increased without limits) the velocity at any point is inversely as the perpendicular to the tan-





gent at that point. That is,  $V \propto \frac{1}{p}$ ;  $p$  denoting the perpendicular.

**PROP. 55.**

155. The angular velocity at any point in the curve, is inversely as the square of the distance of the revolving body, from the centre of attraction; that is, inversely as the square of the radius vector at that point.

*Fig. 32.* Let  $AB$ , and  $CD$ , be arcs described in equal intervals of time around the centre  $S$ . then the triangles  $ABS$ ,  $CDS$ , are equal, (by prop. 52;) and when the arcs are diminished without limits, the perpendiculars  $BP$ , and  $DQ$ , will coincide with arcs described around  $S$  with the radii  $SB$  and  $SD$ .

Now, the magnitude of the angle is as the arc directly, and the radius inversely.

Therefore, ultimately the angle  $ASB$  : angle  $CSD$  ::  $\frac{BP}{SB} : \frac{DQ}{SD}$ .

And because the triangles  $SAB$ ,  $SCD$ , are equal, and  $SA$ ,  $SC$ , are ultimately equal to  $SB$ ,  $SD$ ; it follows, (the bases of equal  $\Delta$ s being inversely as the perpendiculars) that  $BP : DQ ::$

$$\frac{1}{SA} : \frac{1}{SC} :: \frac{1}{SB} : \frac{1}{SD}.$$

Consequently, the angle  $ASB$  : angle  $CSD$  ::  $\frac{1}{SB^2} : \frac{1}{SD^2} :: SD^2 : SB^2$ .

**PROP. 56.**

156. The central forces by which a body is retained in its orbit, are as the versed sines of indefinitely small arcs described in equal times.

*Fig. 32.* For, the small arc  $AB$ , may be considered as the diagonal of the parallelogram  $PT$ , of which  $AT$  expresses the momentum at  $A$  commonly called the projectile force, and  $AP$ , the central force.

Hence, the central force at  $B$  : central force at  $D$  ::  $AP : CQ$ .

**PROP. 57.**

157. If a body move uniformly in a circle, the centre of attraction is the centre of the circle.

For, since the sectors described by the radius vector in equal times are equal, and the arcs on which they stand are also equal, the radius vector is every where the radius of the circle. Hence, the centre of attraction is the centre of the circle.

**PROP. 58.**

158. If a body revolve in a circular orbit, the central force acting at any point in the curve, will be directly as the square of the velocity, and inversely as the deflective chord.

*Fig. 33.* Let  $AC$  be the arc described in an instant, and  $S$  the centre of attraction; through the points  $A, S$ , draw  $ASE$ , the deflective chord. Join  $AC, CE$ , and draw  $CB$ , parallel to the tangent at  $A$ .

The triangles  $ABC, ERC$ , are similar (by Euclid, 32, 3). Therefore,  $AB : AC :: AC : AE$ . Therefore,  $AB = \frac{AC^2}{AE}$ .

But ultimately, the chord  $AC =$  the arc, and consequently  $\frac{v^2}{c}$  the velocity in the curve. Hence,  $f = \frac{v^2}{c} \doteq \frac{v^2}{c}$ ,  $c$  denoting the deflective chord.

It is indifferent, whether  $\frac{v^2}{c}$  or  $\frac{2v^2}{c}$  be taken as the measure

of  $f$ . For, a central force, being constant for an instant, may be measured either by the deflection it produces in an instant, or by the velocity which it would generate in an instant, which is as twice the deflection. But, whether  $\frac{v^2}{c}$  or  $\frac{2v^2}{c}$  is taken as the

measure in any calculation, the same measure must always be adhered to.

**COR.** All the curves, which can be described by the action of finite deflecting forces, are of such a nature that we can describe a circle through any point, having the same tangent and the same curvature which the curve has in that point, and which therefore ultimately coalesces with it. This being the case, it is plain that a planet while passing through a point of its orbit and describing an indefinitely small arc, is in the same condition, as if describing the coincident arc of the equicurve circle. Hence, we obtain this most general proposition, that, *the transverse force by which a planet is made to describe any curve, is directly as the square of its velocity, and inversely as the deflective chord of the equicurve circle.*





## PROP. 59.

159. If a body move uniformly in a circle, the central force will be directly as the square of the velocity, and inversely as the diameter or radius.

For in this case, the centre of the circle will be the centre of attraction, (by prop. 57,) and hence, the deflective chord will be the diameter. Hence it follows (by prop. 58) that  $f \propto \frac{v^2}{r}$ .

COR. Hence, when  $v$  is the same  $f \propto \frac{1}{r}$  and when  $r$  is the same,  $f \propto v^2$ . Both these conditions are proved experimentally by the whirling table.

## PROP. 60.

160. If a body move uniformly in a circle, the central force will be directly as the radius, and inversely as the square of the periodic time.

Let  $c$  be the circle described in the time  $t$ , with the velocity  $v$ . Since, in uniform motions  $v \propto \frac{s}{t}$  so  $v \propto \frac{c}{t}$ , but  $c \propto r \therefore v \propto \frac{r}{t}$  and  $v^2 \propto \frac{r^2}{t^2}$ . Now  $f \propto \frac{v^2}{r}$  (by prop. 59). Therefore,  $f \propto \frac{r^2}{t^2} \div r \propto \frac{r}{t^2}$ .

COR. Hence, when  $r$  is the same  $f \propto \frac{1}{t^2}$ , and when  $t$  is the same,  $f \propto r$ . These conditions also, are proved by the whirling table.

## PROP. 61.

161. The velocity in any point of a curve is equal to the velocity, which would be acquired by descending through one fourth of the deflective chord of the equicurve circle by the uniform action of the centripetal force.

**Fig. 33.** For,  $2AB$  is the velocity which would be acquired by descending through  $AB$ , while the body describes  $AC$ . Therefore, this deflective velocity, is to the velocity in the curve, as  $2AB$ , is to  $AC$ . Let  $x$  denote the space through which the body would move, in order to acquire the velocity  $AC$ . Then since in uniformly accelerated motions, the spaces are as the squares of the velocities.

$$\overline{2AB}^2 : AC^2 :: AB : x; \text{ but } AB : AC :: AC : AE; \text{ or}$$

$$2AB : AC :: AC : \frac{AE}{2}; \text{ therefore}$$

$$\overline{2AB}^2 : AC^2 :: 2AB : \frac{AE}{2}; \text{ therefore ex æquali,}$$

$$2AB : AB :: \frac{AE}{2} : x. \text{ Therefore,}$$

$$AB : AB :: \frac{AE}{4} : x, \text{ that is } \frac{AE}{4} = x.$$

**Remark.** If the velocity increase, the chord of the equicurve circle must increase; that is, the path become less incurvated. If the force be increased, the curvature will also increase; for the chord of curvature will be less.

**COR.** Hence, if a body move uniformly in a circle, the velocity will be equal to that which it would acquire by descending through  $\frac{1}{2}$  the radius with the central force constant.

162. Certain facts, deduced by Kepler from a careful examination of the actual motions of the planets, are admitted as fundamental, in applying the principles of Dynamics to the full determination of the theory of planetary motion. These facts, which are three in number, are usually called *Kepler's laws*, and may be expressed as follows :

1. *The right line joining the sun, and any planet describes areas proportional to the times.*

2. *Each primary planet describes an ellipse, having the sun in one focus.*

3. *The squares of the periodical times of the different planets are proportional to the cubes of their mean distances from the sun.*

It has been demonstrated (prop. 53) that, if the areas described by the radius vector around any point are all in one plane, and proportional to the times, that point is the centre





of attraction. Hence it would appear, from the first law, that the planets are retained in their orbits, by a force continually directed to the sun. It is to be noted, however, that the first law of Kepler, is not mathematically true; and that it is not the centre of the sun, but the centre of gravity of the solar system, towards which the forces deflecting the planets are directed. Yet, as the distance of the sun's centre from the common centre of the system is never greater than one of his diameters, the law as announced above, does not deviate much from the truth.

PROP. 62.

163. The centripetal force in any point of an orbit is inversely as the product of the defective chord of the equicurve circle and the square of the perpendicular drawn to the tangent at that point.

For (by prop. 58,)  $f \div \frac{v^2}{c}$  and (by prop. 54,)  $v \div \frac{1}{p}$ .

Therefore, . . .  $v^2 \div \frac{1}{p^2}$ , therefore  $f \div \frac{1}{p^2} \div c$ .

That is . . .  $f \div \frac{1}{p^2 \times c}$ .

PROP. 63.

164. The centripetal force by which a planet is retained in its orbit, is, in any point, as the square of the distance of that point from the focus, inversely.

*Fig. 34.* Let A D B E, be the elliptical orbit of a planet or comet, and S, the focus or centre of attraction. Let A B, be the transverse axis, D E, the conjugate axis, and C the centre. Let P, be any point in the ellipse. Draw the tangent P V; and S N, from the focus, perpendicular to P V. Draw P Q, perpendicular to P N, meeting the transverse axis in Q. Draw Q O parallel to P N, meeting P S, in O. Also, draw Q R perpendicular to P S. Bisect P O, in T.

It is demonstrated in conic sections, that P O is one half the chord of the equicurve or osculating circle, drawn through the

point P. Therefore, P O, is one half the deflective chord, of the planetary orbit. It is also demonstrated, that P R, is one half of the parameter or latus rectum of the transverse axis A B, or that it is a third proportional to A C, and D C. Therefore, P R, is of the same constant magnitude in whatever part of the circumference the point P is taken.

It is evident, that the triangles N S P, R P Q, and Q P O, are all similar, because P N, Q O, are parallel, and S N P, P R Q, P Q O, are right angles.

Therefore,  $PR : PQ :: PQ : PO$ .

Therefore,  $PR : PO :: PR^2 : PQ^2 :: SN^2 : SP^2$ .

Therefore,  $PR \times SP^2 = PO \times SN^2$ . But the latus rectum L, is equal to  $2 PR$ , and the deflective chord c is equal to  $2 PO$ .

Therefore,  $L \times SP^2 = c \times SN^2$ .

But (since by prop. 62  $f \div \frac{1}{c \times p^2}$ )  $f \div \frac{1}{c \times SN^2}$ . Therefore,  $f \div \frac{1}{L \times SP^2}$ ; and since L is a constant quantity,  $f \div \frac{1}{SP^2}$ ; that is, the centripetal force at any point of the orbit, P, is inversely proportional to  $SP^2$ , or to the square of the distance of that point from the focus.

*Remark.* For the reason above mentioned (162) the sun may be safely regarded in ordinary speculations, as the centre of attraction, or as being situated in the focus of the planet's orbit.

#### PROP. 64.

165. If the orbits of the planets be considered as circles, the centripetal forces which retain them severally in their orbits, will be proportional to the squares of their distances from the sun inversely.

For in this case  $f \div \frac{r}{t^2}$ , (by prop. 60.)

But, by Kepler's 3d law, (162,)  $t^2 \div r^3$ .

Therefore, . . . . .  $f \div \frac{r}{r^3} \div \frac{1}{r^2}$ .

That is, the forces are inversely, as the squares of the mean distances from the sun.

But, although the orbits of the planets are nearly circular, they are not accurately so, and those of the comets are very eccentric. Therefore, it is necessary to obtain a more general determination of the law.





## PROP. 65.

166. If a planet describe an ellipse having the sun in one of its foci, and another body describe a circle, whose radius is the mean distance of the planet from the sun, their time of revolution will be the same.

*Fig. 35.* Let  $ABDF$  be the ellipse in which the planet moves, and  $DEG$  the circle (described on  $SD$  the mean distance) in which the other body moves around the centre  $S$ . Let  $DE$  and  $De$  be equal small arcs of the circle and ellipse. Join  $SE, Se$ .

(By conic sections)  $SD$  is one half the chord of the equicurve circle, at the point  $D$  in the ellipse. And (by prop. 61) the velocity in the point  $D$  of the ellipse is equal to that which would be acquired by descending through  $\frac{1}{2} SD$ , and (by cor. to the same prop.) the velocity in the circle  $DEG$ , is equal to that which would be acquired by descending through  $\frac{1}{2} SD$  or half the radius. Therefore, the velocity in the circle is equal to the velocity in the ellipse at  $D$ , and the equal small arcs  $DE, De$ , are described in the same time. Therefore, the areas  $SDE, SDe$ , are described in the same time. But the base  $DE, De$  being equal,  $SDE : SDe :: SD \text{ or } AC : CD$  that is (by conic sections) :: the area of the whole ellipse  $ABDE$  : the area of the circle  $DEG$ . Therefore, the elliptical and circular areas are similar portions of the ellipse and circle; and therefore the times of describing them, are similar portions of the whole times of revolution in the ellipse and circle. Therefore, these revolutions are performed in equal times.

To make this deduction more manifest to the learner, let it be noted that from the above proportion, it follows that there are as many areas equal  $SDE$  in the circle, as there are areas equal  $SDe$  in the ellipse. Therefore, the time of describing the circle : time of describing  $SDE ::$  time of describing the ellipse : time of describing  $SDe$ . Therefore, &c.

167. Hence it follows, that if the planets and comets were projected when at their mean distances from the sun, perpendicularly to the radii vectores, with the velocities which they have in those points of their orbits, they would describe circles round the sun. And as, (by 3rd law of Kepler,) the squares of the periodic times of the planets are as the cubes of their mean distances, and the periods in the circles are equal to the periods in the corresponding ellipses; therefore, the squares of the periodic times in these circles, would be proportional to the cubes of the mean distances, which are

their radii. Hence, it is easy to see, (from prop. 64,) that the centripetal forces at those mean distances, are as the squares of the mean distances, inversely. Hence, (since the gravitation at the *mean* distance in a given ellipse, is to the gravitation at *any other* distance in the *same ellipse* : : as the square of the mean distance : to the square of the other distance inversely; and as this holds of all ellipses;) it follows that the force at any point in one orbit, is to the force at any point in another orbit, inversely as the squares of the distances of those points from the sun; and generally, *that the planets and comets are retained in their orbits by forces which are inversely as the squares of their distances from the sun.*

### III. CENTRE OF INERTIA OF FREE BODIES.

168. The centre of inertia is a point so situated between two or more bodies, that its state of motion, or rest is not affected by their mutual action. Or, it is that point between bodies which still retains the same proportional distance from them, whatever may be their mutual action.

#### PROP. 66.

169. The centre of inertia of two equal bodies, bisects the right line which joins their centres.

*Fig. 36.* Let A and B, be two equal bodies, and C their centre of inertia. They will be equally affected by their mutual action, and describe equal spaces in the same time, as A E, B D, if they attract; or A e, B d, if they repel.

By the definition,  $CA : CB :: CE : CD :: Ce : Cd$ .

Therefore,  $CA : CB :: CA - CE : CB - CD$ .

That is,  $CA : CB :: AE : BD$ .

Therefore,  $CA = CB$  and  $CE = CD$  and  $Ce = Cd$ .

Therefore, C, the centre of inertia, bisects the right line which joins the bodies.

#### PROP. 67.

170. The centre of inertia of two unequal bodies divides the right line, which joins their centres, in the inverse ratio of the masses.





**Fig. 37.** Let A and B, be two unequal bodies, and C their centre of inertia. They will be moved by equal forces, whether they attract or repel; and will describe, in the same time, spaces inversely as their masses. If they attract, let A E, B D, be the spaces described in the same time; and if they repel, A e, B d.

By the definition,  $CA : CB :: CE : CD :: Ce : Cd$ .

Therefore, . . .  $CA : CB :: CA - CE : CB - CD$ .

That is, . . .  $CA : CB :: AE : BD$ .

But, . . .  $B : A :: AE : BD$ .

Therefore, . . .  $CA : CB :: B : A$ .

Therefore, C, the centre of inertia, divides the right line which joins A and B, in the inverse ratio of their masses.

**COR. 1.** The whole distance of the bodies : is to the distance of their centre of inertia from either :: as the sum of both bodies is to the other body.

That is,  $AB : AC :: A + B : B$ .

**COR. 2.** If the number expressing each mass, be multiplied into the number expressing its distance from the centre, the products will be equal.

That is,  $A \times CA = B \times CB$ .

#### PROP.

**171.** If several bodies are situated in the same right line, their common centre of inertia will so divide it, that the sum of the products of the masses and distances on one side = the sum of the similar products on the other side.

**Fig. 38.** Let A, B, D, E, be bodies situated in the right line A E; and C their common centre of inertia.

Then,  $A \times CA + B \times CB = D \times CD + E \times CE$ .

Let S be the centre of inertia of A and B, and s that of D and E. Then  $A \times SA = B \times SB$  (Cor. 2, prop. 67,) and  $SA = CA - CS$ , also  $SB = CS - CB$ .

Consequently,  $A \times CA - CS = B \times CS - CB$

and  $A \times CA - A \times CS = B \times CS - B \times CB$ ;

then by transposition,  $A \times CA + B \times CB = A + B \times CS$ .

In the same manner,  $D \times CD + E \times CE = D + E \times Cs$ .

Hence, if A and B be united at S, and D and E at s, the sum of the products of the masses and distances is still the same. If M, represent A+B, and m represent D+E; then  $M \times CS = m \times Cs$ , (by cor. 2. prop. 67).

Therefore,  $A \times CA + B \times CB = D \times CD + E \times CE$ .

**NOTE.** We have assumed, that A and B, united at S, will have the same action in the system, as when separate at A and B. To prove the propriety of this assumption, let the bodies approach by

their mutual action. Let  $x$ , be the space described by  $S$  in a given time, and  $y$  and  $z$ , the spaces described by  $A$  and  $B$ , in the same time. Then, since  $A \times CA + B \times CB = A + B \times CS$ , always;

$$\dots M \times CS - x = A \times CA - y + D \times CB - z.$$

That is,  $M \times CS - M \times x = A \times CA - A \times y + B \times CB - B \times x$ .

$$\text{But, } \dots M \times CS = A \times CA + B \times CB.$$

$$\text{By subtraction, } \dots M \times x = A \times y + B \times z.$$

That is, the momentum of  $M$ , placed at  $S$ , is equal to the sum of the momenta of  $A$  and  $B$ , at  $A$  and  $B$ . Therefore, the action of the other bodies,  $D$  and  $E$ , on  $M$ , is equal to the sum of their actions on  $A$  and  $B$ . Therefore, the action of  $M =$  the action of  $A + B =$  the action of  $D + E$ , (by 3d law of motion). In the same manner it may be proved, that the action of  $m$  placed at  $s =$  the action of  $D + E =$  the action of  $A + B =$  the action of  $M$ . (w.)

#### PROP. 69.

172. The centre of inertia of three bodies divides the line, which connects any one of them with the centre of inertia of the other two, in the inverse ratio of the mass of the one to the sum of the masses of the other two.

Fig. 39. That is,  $ED : EC :: A + B : D$ , &c.

For, if  $A$  and  $B$ , be supposed to be united at  $C$ , their centre of inertia, their action on  $D$  will remain the same in all respects. Therefore, if all the bodies attract,  $D$  will move in the right line  $DC$ , and the centre of inertia of  $A + B$  and  $D$  will be situated in  $E$ , making  $ED : EC :: A + B : D$ .

#### PROP. 70.

173. Let two free bodies attract, and parallel forces be applied, proportional to their masses and acting the same way; then, their centre of inertia will move in a right line, while the bodies will describe similar curves, and finally meet at that centre.

Fig. 40. Draw  $FG$  parallel to  $AB$ ; and  $CK$ , through the centre  $C$ , parallel to the directions of the forces  $D, E$ . By the action of these forces alone, the bodies would be found at  $F$  and  $G$ , and the centre of inertia  $C$  would be at  $L$ , in the same instant. But, on account of their mutual action, they will not be found at  $F$  and  $G$ , but at some other points, as  $H$  and  $I$ .

The force of attraction by which each body is affected being the same, the directions at every instant will be inversely as the





masses; and therefore the whole deflections, during any finite time from the beginning, will have the same ratio.

Therefore,  $FH : GI :: B : A$ .

But,  $FL : LG :: B : A$ .

Therefore,  $EL - FH : LG - GI :: B : A$ .

That is,  $HL : LI :: B : A$ . Consequently the centre of inertia is at L, (by prop. 67.) In the same manner it may be proved, that in all other positions of A and B, the centre of inertia is in the line CK; it therefore moves in CK parallel to AF and BG.

Because the centre of inertia is always in CK, the bodies will finally meet at some point K of that line. Since the ratio of HL to IL is constant (that is, the ratio of AC : BC,) the curves AHK, BIK, described by A and B, are similar.

174. Let the bodies repel, and all other things as in the proposition. The centre of inertia will still describe a right line CK, parallel to the directions of the forces D, E, and the bodies similar curves bending from that line. The preceding demonstration will shew this, if we substitute repulsions for attractions, and + for—.

#### PROP. 71.

175. If two free bodies attract, and equal parallel forces be applied to them in contrary directions; their centre of inertia being supposed at rest, they will revolve around it in similar curves without disturbing its quiescence.

*Fig. 41.* Let C be the centre of inertia of two bodies A, B; and G, F, the equal parallel forces which act in contrary directions. Then  $AC : BC :: B : A$ ; and since  $G = F$ , the projectile momenta are equal. Hence the projectile velocities are inversely as the masses; and directly as the distances from C; that is,  $vel A : vel B :: B : A :: AC : BC$ .

Through C draw DE meeting the directions of the forces G, F. By similar  $\triangle s$   $AD : BE :: AC : BC :: CD : CE$ . Hence, by the action of G and F, undisturbed by attraction, AD and BE would be described in the same time; and the same point C, would divide the distance DE in the inverse ratio of the masses.

But, while AD, BE, would be described by the projectile velocities, the mutual attraction of the bodies will cause them to move in curves (127). Suppose AH, BI, to be the curves described. The attractions being equal, the deflections will be inversely as the masses.

That is  $DH:EI::B:A$ .

Hence  $DH:EI::AC:BC::DC:EC$ .

Therefore  $DC - HC = DH:EC - IC = EI::HC:IC$ .

Therefore  $HC:IC::B:A$ .

Hence, the same point C still divides the distance HI in the inverse ratio of the masses when the motions are compound. Therefore, the centre of inertia is not disturbed. And the curves are similar. For, all lines drawn from points in AH, through C, to BI, are divided by C in the same ratio, that is the ratio of B to A.

#### ERRATUM.

In the last line of the demonstration of proposition 60: For

" $f \div \frac{r^2}{t^2} \div r \div \frac{r}{t^2}$ ," read " $f \div \frac{r^2}{t^2} \div r \div \frac{r}{t^2}$ ."





## PART III.

### OF THE LAWS OF IMPULSE.

Impulse may be considered as a pressure of very short duration. It is commonly said to be an instantaneous force, because we are incapable of dividing the time of its action into sensible parts. The full effect of impulsion cannot, however, be produced in an *instant*, according to our definition of this term (8.) When a body is moved, from a state of rest, by impulsion, there is a finite velocity produced, if the force be of a finite magnitude. Call this velocity  $V$ . Now, in the instant in which the body quits its state of rest, or  $v = 0$ , it acquires only an infinitely small velocity; and it passes through every intermediate degree of velocity from  $v = 0$  to  $V$ , by the addition of infinitely small increments of velocity, in successive instants or infinitely small increments of time. We cannot conceive it possible to be otherwise. We cannot conceive of a body passing from rest, to the total velocity produced by any finite force, at once.

Indeed, the phenomena of impact and collision, shew that such a saltus does not take place. Bodies suffer by collision a change of configuration. A change of the configuration of a body, implies a motion of its particles, by which they describe finite spaces: and a finite space cannot be described but in a finite time. Since then, there is no such thing as a perfectly hard body, and a change of configuration accompanies every act of impulsion, the full effect of an impulsive force, in changing the state of motion or rest of a body never does take place in an instant.

#### I. DEFINITIONS.

176. When a body in motion comes into apparent contact with another body, either in motion or at rest, the first body is said to strike or impinge against the second. This class of events has been distinguished by the term impact, collision, or percussion.

177. *Impact* is strictly the simple or single action of one body upon another, in apparent contact, by which the state of motion or rest of the latter is changed. It is commonly used, however, to express the collision of bodies. *Point of impact* is the point of nearest contact ; or, that at which the impinging body acts.

178. *Collision* is the reciprocal action of bodies, by impulse. The collision of two bodies is said to be *direct*, when their impulsive forces act in the same right line. The word *percussion* is also used to express the reciprocal action of bodies by impulse ; but perhaps with less propriety.

179. A *perfectly hard* body, is one, whose particles will not change their relative position, when struck or pressed by other bodies. An *imperfectly hard* body, is one, whose particles suffer a minute change of their relative position, by the impact of others. There is no *perfectly hard* body in nature.

180. An *elastic body* is one, whose particles, having suffered a change in their relative position by some external force, exert an inherent force, by which the original condition is restored. An elastic solid recovers its original configuration ; an elastic fluid, its original density, when the changing force is withdrawn.

181. A body is *perfectly elastic*, when it exerts a recovering force, equal in intensity to that which changed its configuration or density. There is no solid in nature perfectly elastic, as the term is applied to this class of bodies. But perhaps, air, water, and other fluids, are perfectly elastic in themselves ; although by a change of their absolute heat, their inherent resisting or recovering force, may appear sometimes greater and sometimes less than the compressing force.

182. *Inelasticity* consists in the absence of that inherent force, by which elastic bodies recover their configuration or density. There is no body perfectly inelastic.





183. When equal parallel forces are applied to every particle of a body, the centre of these forces, that is the place of their resultant, is the centre of inertia or the centre of gravity of the mass. This point is the centre of gravity; for, terrestrial gravitation acts equally on every particle and in parallel directions. By whatever name we may choose to express this point, it is as the centre of parallel forces, principally, that it is of importance in the investigation of the laws of impulse.

184. Every particle of a perfectly hard body is equally moved by a force acting in the direction of the right line which passes from the point of impact through the centre of gravity: Because this line is the direction of the resultant of equal parallel forces applied to all the particles. The ultimate effect of any impulse upon a perfectly elastic solid must be equally distributed among all its particles also, when the direction of the force passes through the centre of gravity. In either case, the force is propagated throughout the mass, by means of the atomic forces, which preserve the connection of the particles during impact.

Hence, when the direction of an impulse, (or indeed of any force,) passes through the centre of gravity of a free body, it will be moved in that direction without any rotation; and the whole of the force will act in the production of motion in the impelled body.

NOTE. Correct the second sentence of art. 178, by saying: *The collision of two bodies is direct when their whole forces act in the right line passing through the centre of gravity of each.*

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## II. OF THE COLLISION OF ELASTIC BODIES.

### PROP. 72.

185. When the impact of a perfectly hard body in motion upon another at rest is direct, the momentum after the stroke will be divided between them, in the direct ratio of the masses.

*Fig. 42.* Let the quantity of matter in the two bodies, A and B, be expressed by A and B. If  $A = B$ , and the former impinge against the latter, then (by 3rd law of motion) A will lose as much momentum as B will acquire. The two bodies will therefore move on in the direction of A's motion, with the whole momentum divided between them.

Let  $A > B$  or  $A < B$ . Still, (by 3rd law of motion) A will lose just as much momentum as B will acquire; and the whole momentum remains the same. While A impels B, and accelerates its motion, B retards that of A. This reciprocal action can exist only while the velocity of A exceeds that of B. The bodies, when their velocities become equal, move on together as one mass. But  $A v =$  the momentum of A; and  $B v =$  the momentum of B. Therefore,  $v$  being the same in each,  $A v : B v :: A : B$ .

**PROP. 73.**

**186.** When the impact of a perfectly hard body in motion, upon another of infinite magnitude at rest, is direct, the impinging body will be at rest after the stroke.

For, since the momentum is divided after impact in the direct ratio of the masses; and the body at rest is infinite; the velocity after impact  $= 0$ .

**COR. 1.** Hence, if the impact be in a perpendicular direction, upon an immoveable plane, or sphere, the impinging body will remain at rest on the plane or sphere, after the stroke.

**COR. 2.** A perpendicular impulse cannot be resolved into two effective forces, having different directions.

**PROP. 74.**

**187.** If a perfectly hard body be projected obliquely on a perfectly hard immoveable plane, the whole momentum will not be destroyed; but, after impact, the body will move along the plane.

*Fig. 43.* Suppose CB, CD, to be two hard immoveable planes placed at right angles, and let the body A be projected in the direction AC in the plane perpendicular to CB and CD. It is evident, that if  $\angle ACB = \angle ACD$ , the force on CB = the force on CD; and each plane will destroy an equal portion of the incident momentum of A, while they acquire equal momenta themselves. Let the angle ACB continually diminish, and the angle ACD augment. Then the impulse on CD will continually augment, and that on CB diminish. When ACB vanishes, there is





no impulse on  $CB$ ; and the greatest which the momentum of  $A$  can impress, is sustained by  $CD$ : and now  $CD$  acquires as much momentum as  $A$  loses.

Hence, there is an effective force, acting in directions perpendicular and parallel to each plane, in every degree of obliquity of impact. It is evident that, impulsive force must be exerted, in the same directions, when the planes are immoveable. It is also evident that, when one of them is removed, and there is an oblique impact on the other, the latter plane will sustain the perpendicular force only, and the body will move forward, with the parallel force unimpaired.

If we substitute spheres for the planes in this demonstration, the oblique impulse will act on them in the same manner as on the planes; that is, perpendicularly at the point of impact, on each sphere.

**COR. 1.** Hence, every oblique impulse may be resolved into at least two effective forces, one perpendicular, the other parallel, to the plane on which the impact takes place; and in the case of spheres, the one perpendicular, the other parallel to the tangent plane, at the point of impact.

**COR. 2.** Hence, if the point of impact be the centre of gravity of a moveable plane, the plane will (by dynamics) move in the direction of the perpendicular force only; as this perpendicular is in the direction of the resultant of all the elementary forces moving the particles of the plane. If the point of impact be not the centre of the plane, that part, towards which the point of impact deviates from the centre, will move with greater velocity; and the momentary axis of motion of the whole plane, will lie out of the plane on the other side of the point of impact.

**COR. 3.** If the perpendicular to the point of impact passes through the centre of gravity of any body at rest, it will move after impact, in the direction of the perpendicular without rotation, (184.)

**COR. 4.** If the perpendicular to the point of impact does not pass through the centre of gravity of the body, it will have a rotatory motion after the stroke.

#### PROP. 75.

188. When a perfectly hard body is projected on a perfectly hard plane, the force or momentum acting on the plane, is to the whole force before impact, as the sine of inclination to radius.

*Fig. 44.* Let  $I$  be the point of impact of a body, projected in the direction  $CI$ , upon the plane  $AB$ . Take, in the direction of the motion, any distance  $DI$ , to represent the force; and resolve it into the directions  $DA$ ,  $DF$ , perpendicular and parallel to the

plane. Then, (prop. 74)  $DA$  or  $FI$ , represents the force which acts on the plane; and  $DF$  or  $AI$ , the parallel force with which the body moves after impact. The  $\angle$ s at  $A$  and  $F$  are right angles, therefore  $DA$  or  $FI : DI :: \sin \angle AID : \text{radius}$ .

**COR.** If the plane be immovable, the velocity of the impinging body after impact : is to the velocity before impact :: as the cosine of the angle of inclination (or as the sine of the angle of incidence contained by the direction of the motion and the perpendicular) : is to radius. For  $DF$  or  $AI : DI :: \cos$  of  $\angle AID : \text{radius}$ .

#### PROP. 76.

189. When both the bodies are spherical, and one moving obliquely, strikes the other at rest; the velocity of the moving body will be to the velocity of the other, after impact, as radius to the cosine of the angle contained by the directions of their motions.

*Fig. 45.* Let  $A$ , moving in the direction  $AC$ , strike  $B$  at the point  $I$ ; and let  $FIC$  be the tangent plane at that point. Take  $AC$  in the direction of  $A$ 's motion before impact to represent its momentum. Resolve  $AC$  into  $AI$  perpendicular, and  $AE$  parallel to the tangent plane. The perpendicular momentum  $AI$  is divided between  $A$  and  $B$ , in the direct ratio of the masses (by prop. 72); and  $AE$  expressing the momentum of  $A$  alone, in the parallel direction, also expresses its velocity in that direction.

Now, let  $AD$  express the common velocity with which  $A$  and  $B$  would proceed in the direction  $AI$ , after impact, by the distribution of the momentum  $AI$ . Then, if we compound the velocity which is common to  $A$  and  $B$  with the velocity  $AE$ , it is evident, that  $AG$ , the diagonal of the rectangular parallelogram  $ADGE$ , will be the proportion and direction of  $A$ 's velocity, and  $AD$ , the proportion and direction of the velocity of  $B$ .

$\angle IAG$  is the angle contained by these directions, and  $\angle ADG$ ,  $\angle AEG$  are right angles. Therefore  $AG : GE$  or  $AD :: \text{radius} : \cos \angle IAG$ . Therefore, after impact,  $\text{vel. } A : \text{vel. } B :: \text{radius} : \cos$  of the angle contained by their directions.

#### PROP. 77.

190. When the collision of two perfectly hard bodies is direct, they will either be at rest after impact, or will move on together, as if they were one mass.

1. Let the momentum of  $A =$  the momentum of  $B$ , and let the directions of the motions be opposite; then the bodies will be at rest after impact. For, there is no resultant of the opposite forces (50).

2. Let the momentum of  $A$  be greater than the momentum of





B, and let the directions be opposite, then there will be a resultant in the direction of A's motion = the difference of the momenta (51). Therefore, A and B will move together in the direction of A's motion.

3. Let both bodies move in the same direction; and let A have a greater velocity than B. Then A overtaking B will accelerate the motion of B: and B at the same time will retard that of A (by 32). These mutual actions can only exist while the velocity of A exceeds that of B. Therefore, when the velocities become equal, the bodies will proceed as one mass with an uniform motion.

PROP. 78.

191. The momentum estimated in one direction is the same before and after impact.

In the first case of the preceding proposition, the momentum estimated in either direction, before and after impact, = 0.

In the second case, let  $d$  = the difference of the absolute momenta of A and B in opposite directions. Then  $A V - B v = +d$ , is the positive momentum in the direction of A's motion, both before and after impact; and  $B v - A V = -d$ , is the negative momentum in the direction of B's motion, both before and after impact.

In the third case, the momenta being in the same direction, B gains as much as A loses during impact, (by the 3rd law of motion). Therefore,  $A V + B v (= A V - \frac{1}{2}d + B v + \frac{1}{2}d)$ , is the momentum in one direction, both before and after impact; and  $-A V + B v$ , is that estimated in the other direction both before and after impact.

PROP. 79.

192. When the collision of two perfectly hard bodies is direct, their common velocity, after impact, is equal to the whole momentum (estimated in the direction of that velocity) before impact, divided by the sum of their masses.

Let A and B be their masses, and V and v their velocities. Then A V and B v will be their momenta before impact. The sum of these momenta =  $A V + B v$ , if the bodies move in the same direction; and =  $A V - B v$ , if in opposite directions; the momentum of A being supposed the greater.

Let C = their common velocity after impact; then  $\overline{A + B} \times C$  = their whole momentum after impact, when they move as one mass. But by the preceding proposition,  $\overline{A + B} \times C = A V \pm B v$ .

Therefore  $C = \frac{A V + B v}{A + B}$ , when the bodies move in the same

direction, before impact; and  $C = \frac{A V - B v}{A + B}$ , when they move in opposite directions.

That is, the common velocity after impact = the whole momentum in the same direction, before impact, divided by the quantity of matter.

### III. OF THE COLLISION OF ELASTIC BODIES.

#### PROP. 80.

193. If a perfectly elastic body be projected on a perfectly hard immoveable plane in a direction perpendicular to its surface, the body will be reflected perpendicularly; and the velocity after impact will be equal to the velocity before impact.

For, the impinging body suffers compression by the re-action of the unyielding plane, as long as it retains any part of its incident momentum. When the whole of that momentum is destroyed, the body begins to recover its original configuration. This recovering force, called the force of restitution, is of equal intensity with the compressing force; but the plane being immoveable, the body, while it recovers its configuration, must recede from it; and the force of restitution is exerted in a direction opposite to the force of compression. The body will, therefore, be reflected from the plane, in the perpendicular to the point of impact, with a velocity equal to its velocity before impact.

If the plane, instead of being perfectly hard, be perfectly elastic, the same thing will happen. For, although the plane will suffer compression by the impulse, the force of restitution being equal to the force of compression, it will re-act on the moving body just as much as if it were perfectly hard.

When an ivory or glass ball falls perpendicularly on a horizontal hard plane, it ascends after impact in the same line, and nearly to the height from which it fell. If the ball were perfectly elastic, and the plane perfectly hard, the former would, in a vacuum, ascend exactly to the height from which it fell.

COR. 1. The perfectly hard immoveable plane sustains, from the impact of a perfectly elastic body, a force twice as great as from that of an inelastic one, when the incident momenta are equal. For, the whole of the incident momentum is destroyed, whether the body be elastic or inelastic. But if perfectly elastic, the force of restitution acts on the plane as much as the incident momentum.

COR. 2. The impinging body is also affected by a double force during impact. For, its incident momentum is destroyed, and an equal momentum is impressed on it in the opposite direction by





the reaction or resistance of the plane, during the exertion of its force of restitution.

COR. 3. Hence, in the collision of two perfectly elastic bodies, the change in the velocity of each, will be twice as great as if they were perfectly hard.

PROP. 81.

194. When a perfectly elastic body is projected obliquely upon a perfectly hard immoveable plane, it is reflected; and the angles which the lines of its motion make with the plane, before and after impact, are equal.

Fig. 46. Let the body C, strike the plane M N at I. From any point D, in the line of incidence, draw D A perpendicular to the plane; and at a distance equal to I A, on the other side of I, draw B F also perpendicular, and equal to D A: join D F and I F: and from the point of impact erect the perpendicular I E; D E, E F, are also equal.

If D I represent the whole motion before impact, then D A or E I will represent the perpendicular motion, and A I or D E the parallel motion, which remains the same. The former is destroyed by impact during the compression of the body; and by the force of restitution, an equal motion in the opposite direction is generated. Hence the motions after impact are I E and I B = A I: these compose the motion I F. Therefore, the body C moves in the direction I F after impact. It is plain from the construction, that the angle B I F = the angle A I D; that is, the angle of reflection = the angle of incidence.

PROP. 82.

195. To find the velocities after impact of any two elastic bodies which move with any velocities before impact.

Let A and B be the bodies, and V, v, their velocities before impact.

1. When the bodies move in the same direction before impact.

When the compressing action terminates, there is for an instant a common velocity C; and because the whole momentum at that

$$\text{instant} = A V + B v, C = \frac{A V + B v}{A + B}.$$

The momentum of A at that instant = A C and that of B = B C. By the action of compression A has lost a portion of its momentum = A V — A C, and B has gained a momentum = B C — B v.

The force of restitution produces another change in each equal to the former.

Therefore, 2 A V — 2 A C = the whole momentum lost by A; and 2 B C — 2 B v = the whole momentum gained by B.

Hence, the momentum of A after impact =  $AV - 2AV - 2AC$   
 $= 2AC - AV$ ; and that of B =  $Bv + 2BC - 2Bv = 2BC$   
 $- Bv$ .

Therefore, the velocity of A after impact =  $2C - V =$

$$\frac{2AV + Bv}{A + B} - V = \frac{2AV + 2Bv - AV - BV}{A + B}$$

$\frac{A - B \times V + 2Bv}{A + B}$ ; and the velocity of B after impact =  $2C$

$$- v = \frac{2AV + Bv}{A + B} - v = \frac{2AV + 2Bv - Av - Bv}{A + B}$$

$$\frac{B - A \times v + 2AV}{A + B}$$

2. When the bodies move in opposite directions.

At the termination of the compressing action the whole momentum =  $AV - Bv$ ; therefore the common velocity  $C = \frac{AV - Bv}{A + B}$

The momentum of A at that instant =  $AC$ ; and that of B =  $BC$ . By the action of compression A has lost a portion of momentum =  $AV - AC$ ; and B has lost its former momentum  $Bv$  and gained a momentum =  $BC$  in the contrary direction.

The force of restitution produces another change equal to the former.

Therefore  $2AV - AC$  = the whole momentum lost by A; and  $2BC + Bv$  = the momentum lost by B together with that gained in the opposite direction.

Hence, the momentum of A after impact =  $AV - 2AV - AC$   
 $= 2AC - AV$ ; and that of B =  $2BC + 2Bv - Bv = 2BC$   
 $+ Bv$ .

Therefore, the velocity of A after impact =  $2C - V =$

$$\frac{2AV - Bv}{A + B} - V = \frac{2AV - 2Bv - AV - BV}{A + B}$$

$\frac{A - B \times V - 2Bv}{A + B}$ ; and that of B =  $2C + v =$

$$\frac{2AV - 2Bv}{A + B} + v = \frac{2AV - 2Bv + AV + Bv}{A + B}$$

$$= \frac{A - B \times v + 2AV}{A + B}$$





A direct examination of a few of the more remarkable cases of the collision of elastic bodies, will, it is believed, be more interesting to the student, than deductions from a general proposition.

196. When two equal bodies meet with equal velocities, they recede after impact with equal velocities; and the velocities after impact are equal to the velocities before impact.

If the bodies were perfectly hard, they would be at rest after impact. But being perfectly elastic, as soon as their incident momenta are destroyed by collision, the force of restitution of each begins to act in the opposite direction. Hence the bodies recede after impact.

And since the force of restitution acts equally on each, and is equal to the compressing force, they recede with equal velocities; and the velocities after impact = the velocities before impact.

197. When one of the two equal bodies is at rest before impact, the impinging body will be at rest after impact; and that which was previously at rest will move on with a velocity equal to the incident velocity of the impinging body.

Let A and B be the equal bodies, the former in motion, and the latter at rest before impact. If they were perfectly hard, they would move on as one mass after the stroke. Their common velocity would be only half the original velocity of A; and this would lose half its velocity, while B would gain as much. Double the change of velocity in each. Then A will lose the whole of its velocity, and B will gain a velocity equal to that whole. This last condition must take place when the bodies are perfectly elastic. (Prop. 80, Cor. 3.)

198. If two equal bodies move in one direction with unequal velocities, their velocities will be interchanged by collision.

Let A move with the greater velocity  $V$ , and overtake B moving with the less velocity  $v$ . Let the difference of their velocities before impact  $(V - v) = d$ . At the termination of the compressing action, A's velocity is diminished  $\frac{1}{2}d$ , and B's velocity is augmented  $\frac{1}{2}d$ , (33.) But the force of restitution causes another change =  $\frac{1}{2}d$  in the velocity of each body. The whole retardation of A's motion is, therefore, equal to the difference  $d$ , and the whole acceleration of B's motion is equal to  $d$  also.

Therefore, since  $V - d = v$ , and  $v + d = V$ , the bodies move after impact with interchanged velocities.

199. If two equal bodies move in opposite directions with unequal velocities, their velocities will be interchanged by collision.

Let A and B meet with the velocities  $V$  and  $v$ . When the compressing action terminates, there is for an instant a common velocity  $= \frac{1}{2} \overline{V - v} = \frac{1}{2} d$ , in the direction of A's motion. Consequently, at this instant, the change in the velocity of each  $= v + \frac{1}{2} d$ ; and the force of restitution produces another change on each  $= v + \frac{1}{2} d$ .

Therefore B recedes after the impact with a velocity  $= \overline{v + \frac{1}{2} d} + \frac{1}{2} d = v + d = V$ .

And A recedes after impact with a velocity  $= \overline{v + \frac{1}{2} d} - \frac{1}{2} d = v$ ; for  $V = v + d$ ; and  $v + d - v + \frac{1}{2} d = \frac{1}{2} d$  = the remainder of A's incident velocity. But the change by restitution  $= \overline{v + \frac{1}{2} d}$  destroys that remainder and gives a velocity of retrocession  $= v$ .

200. If two bodies meet with equal velocities, and one of them contain three times the matter of the other, the greater body will remain at rest after impact, and the less will be reflected back with twice its former velocity.

Let  $A = 3 B$ , and let  $V$  be the velocity of each. Then  $3 B V$  = the momentum of A, and  $B V$  = the momentum of B. When the compressing action terminates, there is for an instant a common velocity; and because the whole momentum at this instant  $= 3 B V - B V = 2 B V$ , the common velocity =

$$\frac{2 B V}{3 B + B} = \frac{2 B V}{A + B} = \frac{2 B V}{4 B} = \frac{1}{2} V.$$

Therefore, the momentum of A, at this instant,  $= \frac{1}{2} A V$ , and that of B  $= \frac{1}{2} B V$ .

The force of restitution produces another change on each, equal to the former, so that the remaining half of A's momentum is destroyed, and a momentum equal to that half is added to B's momentum. Therefore, after impact A remains at rest; and B is reflected with a momentum  $= \frac{1}{2} A V + \frac{1}{2} B V = 2 B V$ , or

$$\text{with a velocity} = \frac{2 B V}{B} = 2 V.$$





201. When a ball in motion strikes the nearest of a row of equal balls placed in sensible contact in a right line, the stroke being direct, the impinging ball remains at rest after the stroke, and the last of the row flies off with a velocity equal to that of the former before impact.

An interchange of the velocities  $V$  and  $v = 0$  first takes place between the impinging ball and the nearest of the row. This first ball moves through the insensible space between it and the second, with the velocity  $V$ , while the impinging ball has its velocity reduced to  $v = 0$ , and is at rest. The second and third interchange velocities in like manner, and so on successively. Each ball moves through the insensible space between it and the next with the velocity  $V$ ; impinges on the next, and has its velocity reduced to  $v = 0$ ; and the last ball only flies off.

In like manner, if two balls strike, two will fly off; if three strike, three will fly off; and so of any greater number.

202. When a smaller body in motion strikes a greater at rest, the smaller is reflected with a less velocity, and the greater advances, also with a less velocity.

Let  $A$  be greater than  $B$  whose velocity is  $V$ . If  $A$  were infinite,  $B$  would be reflected with a velocity equal to  $V$ . If  $A$  were equal to  $B$ , the latter would remain at rest after impact. Therefore, since  $A$  is greater than  $B$ , yet less than infinite,  $B$  will be reflected with a velocity less than  $V$ .

Again, if  $A$  were equal to  $B$ , the velocity of  $A$  after impact would  $= V$ ; if  $A$  were infinite, its velocity after impact would  $= 0$ . Therefore, since  $A$  is greater than  $B$ , yet less than infinite, it advances after the stroke with a velocity less than  $V$ .

203. The momentum communicated by a smaller body, to a larger at rest, is greater than its own incident momentum.

If the bodies were equal, the striking one would remain at rest, and the other would acquire an equal momentum after impact. But when the body at rest is greater, the other is reflected. The body previously at rest must, therefore, acquire a momentum equal to the sum of the incident and reflected momenta of that which gave the stroke. (This is evident from the reasoning in 193.)

Hence, when the body at rest is exceedingly great, compared with the impinging body, the momentum of the former

after impact, nearly doubles that of the latter before impact : and the momentum of the impinging body after impact, is nearly equal to its momentum before impact.

Hence, likewise, when the body at rest is infinite, the sum of the momenta after impact, is triple the momentum before impact. But the sum of the velocities after impact, is only equal to the velocity before impact.

204. When a larger body in motion strikes a smaller at rest, it communicates to the smaller only as much momentum as it loses. The larger still proceeds with a less velocity, and the smaller moves in the same direction with a greater velocity.

If A in motion = B at rest, before impact, then A will be at rest, and B will move after the stroke with the former velocity of A ; and the momenta before and after the stroke will be equal. But if  $A > B$ , the excess into the velocity before impact ; that is,  $D \times V$  may be considered as a momentum which is only diminished in proportion to an augmentation of motion which it communicates to B, (30.) Hence, the whole of A's loss of momentum = B's gain ; and A advances with a less, and B with a greater velocity than that of the former before impact.

#### IV. OF IMPULSIVE ENERGY.

205. The term energy has been employed to express the efficacy of a force, in producing certain mechanical effects, which are as the product of the quantity of matter into the square of the velocity.

When a rammer strikes a pile with a given velocity, it will cause it to penetrate to a certain depth ; but, if the velocity of the rammer were double, the depth to which the pile would penetrate would be quadruple, with due allowance for the mass of the pile which is moved. A ball moving with any velocity, and entering a soft substance as tallow, would penetrate four times as far if the velocity were doubled. If an elastic ball strike a horizontal plane with any velocity, and another strike it with a double, the





latter will rise after reflection to four times the height of the former; for, the velocity of each ball after impact, is equal to the velocity before, and by the laws of uniformly retarded motions, the heights to which the bodies will ascend after reflection, must be as the squares of their velocities. The double velocity requires a double time for its extinction by the same retarding force, whatever be the nature of this force, provided its action be uniform. The soft earth into which a pile is impelled, and the tal-  
low into which a heavy body falls, are nearly such. Hence, the mechanical effect of the force is nearly as the square of the velocity of the agent. In general, the mechanical action of any machine in motion is proportional to the square of the velocity, when theoretically investigated; but, there are various causes of defalcation in practice, the principal of which are friction, flexibility, and compressibility of the parts of the machine. It is probable, that if the face of the rammer were covered with moist cork, the pile would not be driven. A steel wedge, driven by a steel hammer, will, on the other hand, give an effect which far exceeds that produced by an iron hammer on an iron wedge, when all other circumstances are the same.

206. The relative velocity of two bodies, is the velocity by which their distance is augmented or diminished. Either of the bodies may be considered as moving with this velocity, relatively to the other; indeed, impact and every mutual action, is influenced by the relative velocity only.

#### PROP. 82.

207. The sum of the energies of two perfectly elastic bodies, after impact, is equal to the sum of their energies before impact. That is, the sum of the products of each mass, into the square of its velocity, before, and after impact, is the same.

Let the bodies A and B have a relative velocity; then, their velocities towards the centre of inertia will be inversely as their masses, and their momenta in opposite directions will be  $A \times B$  and  $B \times A$ . Now, if the centre of inertia have also a motion C with respect to a quiescent space, in the direction of A, the velocities will be  $C + B$  and  $C - A$ , respectively, and the joint energies will be  $A \times (C + B)^2 + B \times (C - A)^2 = \int x$ .

But after collision, the velocities of B and A, relative to the centre of inertia, are in a contrary direction, the motion of that

centre remaining the same (168.) Therefore, the velocities are  $C - B$  and  $C + A$  respectively, and the energies  $A \times (C - B)^2 + B \times (C + A)^2 = \int y$ .

To see that  $\int x = \int y$ , observe,

Before impact, energy of  $A = A \times (C + B)^2$

After impact, energy of  $A = A \times (C - B)^2$

The difference  $= A \times (C + B)^2 - A \times (C - B)^2$

Also, before impact, energy of  $B = B \times (C - A)^2$

After impact, energy of  $B = B \times (C + A)^2$

The difference  $= B \times (C + A)^2 - B \times (C - A)^2$

By the help of the 2d Prop. 4th Book, Euclid, it may be seen that each of these differences is  $= 4 A B C$ ; the differences are therefore equal. Therefore, the energy gained by one of the bodies ( $B$ ), is equal to that lost by the other ( $A$ )  $\therefore \int y = \int x$ .

The equality of  $\int x$  and  $\int y$  may be proved more briefly without recurring to the procedure of nature. For, by transposition, we get two quantities, each of which is equal to a third.

Viz:  $A \times (C + B)^2 + A \times (C - B)^2 (= 4 A B C) = B \times (C - A)^2 + B \times (C + A)^2$ . Therefore, the quantities expressed by the original order of the terms are equal.

208. This proposition is a particular case of the famous *conservatio virium vivarum*, regarded by many philosophers of the last century, as a fundamental principle in mechanical action, as a necessary law of nature, and as the proper measure of force. Although this last opinion is universally abandoned, the efficacy of a force, in producing effects which are proportional to  $q v^2$ , well deserves a distinct denomination.

It has been shewn that the sum of the momenta before and after collision, reduced to one direction, is the same. This general law of mechanical action, has been called the *conservatio momentorum*. The energies are estimated in the respective directions of the velocities before and after collision: It is when thus estimated, that the same quantity of energy is preserved, and it is the absolute quantity.





## PROP. 84.

209. In the collision of inelastic bodies, there is a loss of energy by impact, and if perfectly inelastic, the loss is proportional to the square of the relative velocity.

Before impact the sum of the energies  $= A V^2 + B v^2 = \int x$ .

The common velocity after impact, when the bodies move the same way  $= \frac{A V + B v}{A + B}$ . Therefore, the whole energy after

$$\text{impact} = (A + B) \times \frac{(A V + B v)^2}{(A + B)^2} = \frac{(A V + B v)^2}{A + B} =$$

$$(\text{by 4.2. Euclid}) \frac{A^2 \times V^2 + B^2 \times v^2 + A B \times 2 V v}{A + B} = \int y.$$

$$x \times (A + B) = A^2 \times V^2 + B^2 \times v^2 + A B \times V^2 + v^2$$

$$y \times (A + B) = A^2 \times V^2 + B^2 \times v^2 + A B \times 2 V v.$$

$$\text{The difference} \quad . \quad . \quad . \quad . \quad A B \times (V - v)^2$$

$$\text{Therefore, loss of energy} = \frac{A \times B}{A + B} \times (V - v)^2$$

Since  $A, B$  are constant, the loss of energy  $\propto (V - v)^2$ ; or it is proportional to the square of the relative velocity  $V - v$ .

When the bodies move in opposite directions,  $y \times (A + B) = A^2 \times V^2 + B^2 \times v^2 - A B \times 2 V v$ .

The difference from  $x = A B + (V + v)^2 \propto (V + v)^2$ , and this is the energy lost, which is proportional to the square of the relative velocity  $V + v$ .

## V. OF THE ROTATORY EFFECTS OF IMPULSION.

210. Two points are relatively in motion, when their distance varies on account of a real motion, either of one, or of

both of them : and they are relatively at rest, whatever may be their real motions, when their rectilinear distance continues the same.

211. When a moving point describes a circle in a quiescent plane, the only point within the circumference, with which it is relatively at rest, is the centre : every other point, is in relative motion. The centre of the circle, has, therefore, a peculiar relation to the moving point. If the moving point and the centre, be supposed to be connected by a right line, every other point in the quiescent plane, is in relative motion with respect to this line. The centre of the circle is, therefore, properly called the centre of motion. When the whole area of a circle is in motion about its centre, this point is at rest, and all others in motion, with respect to the surrounding space, which is supposed to be quiescent. If one point in a right line describe a circle, and another point in it is at rest with respect to the plane of the circle, all other points of the line, describe concentric circles.

When a surface or solid moves about a quiescent line, every point in the moving surface or solid, describes a circle, of which, the centre lies in the quiescent line.

This kind of motion is called rotation. The quiescent point, about which the others move, describing circles in the same plane, is called the centre of rotation : the quiescent line, about which a surface or solid moves, is called the axis of rotation.

212. The centre or axis of rotation is necessarily quiescent with respect to the points moving round it (210) ; but it may be in motion with respect to some surface or space, which is considered as at rest. For instance, when a circle rolls along a quiescent line, the centre retains the same invariable position, with respect to any point in its surface, and is therefore at rest with respect to it : but, while this point in the surface, approaches or recedes from the line, the centre advances in a parallel direction ; and is therefore in motion with respect to any point taken in the quiescent line.





This combination of a rotatory and a progressive motion, is familiar in the operation of carriage wheels, rollers, &c. It is a subject, the investigation of which, in its whole extent, is esteemed the most difficult in mechanical philosophy ; yet, the application of this combination, is of very extensive utility in terrestrial mechanics ; and some acquaintance with its principles, is necessary in order to understand the sublime theories of physical astronomy. In this arrangement, there can be admitted only a few of the more simple conditions of this interesting combination of motions.

PROP. 85.

213. If equal impulses act simultaneously, and in contrary directions, on two quiescent bodies invariably connected by a right line, the bodies will move in concentric circles about their centre of inertia, which will remain at rest.

*Fig. 47.* Let *A* and *B* be the bodies ; *G* their centre of inertia ; *f* and *f* the equal forces. With the radii *A G*, *B G*, describe the concentric circles *A a L*, *B D b* ; and through *G* draw *a b*, *C D* ; also, construct the rectangular parallelograms *A m C E*, *B n D F*, about the similar arcs, *A C*, *B D*.

First ; let the forces *f*, *f*, act perpendicularly to *A B*. The momenta of *A* and *B* will be equal, and their velocities inversely as the masses ; and since, by similar figures, *A m* : *A G* : *B* :: *B n* : *B G* : *A*, the spaces *A m* and *B n* would be described in the same time, if the bodies were free and uninfluenced by any deflective force. But the cohesion of the invariable line *A B* is a force which affects the two bodies equally, and in opposite directions, since it is a mutual force between every two particles. Therefore, the deflections are inversely as the masses. And *C m* : *A m* : *B* :: *D n* : *B n* : *A*. Consequently, the paths in which *A* and *B* rotate, coincide with the concentric circles *A a L*, *B D b* ; and their centre of inertia *G*, which coincides with the centre of those circles, remains at rest.

Again, let the parallel forces be inclined to *A B*. Resolve each into two, the one perpendicular and the other parallel to *A B*. The latter, being equal and opposite, destroy each other ; the former being also equal, parallel, and acting contrary ways, will, as shewn above, cause the bodies to describe concentric circles.

## PROP. 86.

214. If equal parallel impulses act in contrary directions, on two connected bodies already in progressive motion, neither the velocity nor path of the centre of inertia will be changed by their action.

For, by the preceding proposition, equal parallel forces applied in contrary directions, have no moving action on the centre of inertia. Consequently, they can neither accelerate, nor retard, nor deflect its existing motion.

This is a sufficient demonstration of the proposition. But to assist the conception of the student, let  $A C$ ,  $B D$ , be the similar arcs which the parallel impulses would cause the bodies to describe in any portion of time from a state of rest; and let  $G H$  be the space which would be described by the existing progressive motion of the centre  $G$  in the same time, then, by the combination of these motions,  $A B$ , instead of the parallel attitude  $I K$ , will be in the attitude  $c d$ , at the end of that time, making the arc  $I c = A C$ , and  $K d = B D$ , and the distance  $G H$  unchanged.

COR. 1. Hence, the actual rotation will be the same, whether the bodies were previously at rest or in motion; that is, the quantity of rotatory motion of every part will be the same in either case.

COR. 2. Hence, also, the whole progressive motion of the system, that is, the total momentum estimated in the direction of the motion of the centre of inertia, will be the same before and after the application of the impulses.

COR. 3. Since the centre of inertia still continues in the direction of its original progressive motion, it suffers no rotatory motion. The rotation is, therefore, performed about this point.

## PROP. 87.

215. If an impulse be applied in a perpendicular direction to a material inflexible line, at any point but the centre, a uniform rotatory motion will be produced.

*Fig. 48.* Let the force  $F$  be applied at  $P$ . Then, the extremity  $B$  will move with greater velocity than the extremity  $A$ ; and the line  $A B$  will rotate about some point between  $A$  and  $B$ . Suppose  $f, f$ , to be forces simultaneously applied at  $A$  and  $B$ , acting the contrary way; and suppose  $f + f = F$ ; also,  $f : f :: B P : A P$ . Then the line  $A B$  will be at rest, and the action of  $F$  on every point of it will be counterbalanced, (71.) Therefore, the





action of  $F$  at the point  $A = f$ , and the action of  $F$  at the point  $B = f$ .

Therefore, the action of  $F$  at  $A$  : the action of  $F$  at  $B$  ::  $BP$  :  $AP$ . Therefore, when  $F$  alone acts, the vel.  $A$  : vel.  $B$  ::  $f$  :  $f$  ::  $BP$  :  $AP$ .

Let  $A m$ ,  $B n$ , be small arcs sensibly coinciding in direction with the perpendicular right lines  $A a$ ,  $B b$ , which are equal and parallel to  $G I$ ; and let  $A m : B n :: BP : AP :: f : f$ . Then it is plain, that  $A m$  and  $B n$  will be described in the same minute interval of time, and that  $m n$  will be the attitude of the line  $AB$  at the end of that interval. Now, since the velocities of the points  $A$  and  $B$ , undisturbed by other forces, will continue uniform, whatever deviation from the parallel position ( $a b$ ) may exist at the end of the first interval, there will be a double deviation at the end of the second equal interval, and so on continually.

#### PROP. 88.

216. If an impulse be applied to a right line, as in the preceding proposition, the centre of inertia will move with the same progressive velocity as if the force acted directly at this point.

By the action of  $F$  at the centre  $G$ , the points  $A$ ,  $G$ ,  $B$  would describe equal parallel lines as  $A a$ ,  $G I$ ,  $B b$ , in some small interval of time; and the velocities would continue equal. Now, suppose  $F$  to act at any other point  $P$ , and let  $f, f$ , be the elementary equivalents of this force. Then  $f : f : F :: BP : AP : A B$ . Since these equivalents will produce the same effect as  $F$ , they may be substituted for this force. Therefore,  $A m : B n (= BP : AP = f : f)$  will be the ratio of the spaces described in the same interval by the action of  $F$  at the point  $P$ . Since  $G$  is the middle of the line  $AB$ ,  $A m + B n = 2 G I$ , or  $G I$  is an arithmetical mean to  $A m$  and  $B n$ .

Therefore,  $G I$  will be described exactly in the same time, whether  $F$  acts at  $G$  or at  $P$ .

COR. 1. The centre of rotation coincides with the centre of inertia. For, since  $A m$ ,  $G I$ ,  $B n$ , are in arithmetical proportion, and  $A a$  and  $B b$  equal, the small arcs  $m a$ ,  $n b$ , which measure the velocities of rotation of the points  $A$  and  $B$ , are equal. But equal arcs, described by points of the same inflexible line in the same interval of time, must be described with equal radii. Therefore,  $G$  is the centre of rotatory motion.

COR. 2. The centre of inertia will describe a right line. For, since  $m a = n b$ , the velocities of rotation with which  $A$  and  $B$  recede from the parallel attitude, are equal : suppose  $F$  to act at  $G$ , and that the velocities of rotation are produced by equal paral-

lel forces, acting in contrary directions to each other, at A and B: these equivalent forces will not cause any deflection from the rectilinear path G I (214.) But, when F alone acts at P, the whole effect is the same; therefore, the centre of inertia will move in a right line.

COR. 3. If A and B be equal particles, or equal masses, whose centres are joined by the invariable right line A B, it is plain that whatever has been shewn in this proposition and corollaries concerning the rotation of a right line, will be true of the system. And if A and B be unequal masses, the same is still true. This may easily be seen by conceiving the impulse F to be applied at G, the centre of inertia of the system, and conceiving the equivalents  $f, f$ , to act in contrary parallel directions, so as to produce the same rotatory effect which would arise from the action of F alone, if applied at any other point P. Then we have a progressive motion joined with the rotatory effect of  $f, f$ , and all the conclusions of 214 and 216 are obviously derived.

217. When the system consists of several connected bodies in different planes, or of an irregular mass, there is a complication which renders the subject of rotation very intricate. Yet it can be demonstrated, that every particle, lying in the same plane with the direction of the impulse and the centre of inertia, rotates about this centre; . . . that, all the particles of the system rotate about a line or axis passing through the centre of inertia perpendicularly to that plane; . . . that, if the whole force of the impulse be expended on the system, the quantity of progressive motion will be the same as if no rotation were produced, and the quantity of rotatory motion will be the same as if the centre of inertia were to remain quiescent; . . . also, that the velocity of the centre of inertia will be the same as if the impulse were direct, and that it will describe a right line.

The angular velocity of every particle of the system A B, whatever may be the arrangement of its parts, is the same. For, every particle is supposed to be invariably connected with the axis of rotation; and, but for the existence of progressive motion, all would describe circles about this axis in the same time; (see spontaneous centre of conversion.)

The subject of rotatory motion is further illustrated in P. IV. II.





## PART IV.

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### OF THE LAWS OF PRESSURE.

Under this head, are arranged those elementary principles of the equilibrium and pressure of solid bodies, in their proximate action, with which the student should be familiar, before he enters upon the study of any elaborate or abstruse work on Mechanics.

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#### I. OF PRESSURE IN GENERAL.

218. The reader is already aware, that a pressure, in the general acceptation of the term, is a force of sensible duration; and that it may be of the nature, either of an attraction, or a repulsion. He also knows, that a pressure may be exerted, either as a *moving force* generating velocity, momentum, energy, or as a resisting force, preventing other powers from producing these effects or conditions. It is very obvious, that a pressure may be fully exerted, and yet not generate motion. In this case, it may, very properly, be termed a quiescent pressure or a quiescent force; as that, which generates motion, is called a moving force.

Quiescent pressures, exerted by masses, in physical contact, or connected by cohering matter, as a book resting on the table, or a suspended body tending from its support, have been called *dead pressures*. This name was probably intended to convey the idea, that their exertion is unattended with that mechanical efficacy, which was called *living force*,

and which has lately received the more appropriate name of *energy*.

It will be seen presently, that the proximate agent in all *dead pressures*, is a corpuscular power, which acts only at insensible distances; and that gravitation and other forces acting at sensible distances, operate merely as *occasional causes*, by placing masses or their particles within the limit of corpuscular action.

219. When the particles of a mass, are united by cohesion, they may be considered as placed at their natural unrestrained distances. If any foreign force cause them to change their distances without altogether destroying the order of aggregation, they will tend to resume their former state of proximity, as soon as the disturbing cause is removed: if they had been made to recede, corpuscular attraction will restore them to their former places; if they had been made to approach, corpuscular repulsion will separate them to their original distances. This is what constitutes the elasticity of masses. An elastic string may be extended beyond its natural length, by some foreign force; but when this force is removed, it contracts again. An elastic solid, may be variously compressed, either through its whole extent, or in certain parts; but, upon the removal of the compressing force, it recovers its former state. Inelastic bodies, are such as suffer from comparatively small forces, a radical change in the order of the aggregation of their particles; or else they are supposed incapable (as perfectly hard bodies) of being extended or compressed by any finite force. Nevertheless, all solids in nature, are more or less elastic; and their particles are arranged in the limits of two opposite corpuscular forces. At a greater distance, the particles would mutually attract; this constitutes their cohesion: at a less distance, they would repel; this constitutes their mutual impenetrability.

220. Hence, if a row of particles were arranged in a right line at their natural distances, and if a pressure were applied to the particle at one end of the row, in the direction of the line, it will appear to act on them all, and to impel the whole





row. But, this is not the process of nature. The external force moves only the first particle. This particle is now within the limit of the mutual repulsion between it and the second. This inherent force moves the second. The second and third are now within the limit of their mutual repulsion, and this inherent force moves the third. The inherent repulsive forces of the particles of the row are thus successively brought into action, each one repelling that next to it; and always in the direction of the foreign force and with an equal intensity. Now, it is obvious, that the force which moves the second particle, is not the identical pressure which moved the first; nor are any two particles moved by the same identical force, but each exerts its inherent force of repulsion on the next. Just so the first of a row of equal ivory balls in physical contact being struck, each ball moves through an insensible space, and comes within the limit of repulsion of the next; the repulsion being mutual, each impels the next, and being itself equally repelled, is brought to rest. Strictly, then, a particular impulse or pressure cannot be propagated through a mass; but a succession of quiescent forces are brought into action, and thus motion is generated in every particle in the same direction, and the whole system advances.

If the external force act in the contrary direction, tending to draw the particle to which it is applied from that next to it, the distance between them is increased, and a mutual attraction is exerted. This force causes the second particle to follow the first. The distance between the second and third is now increased, and they attract; the third particle follows the second. In this way, the mutual force of cohesion causes one particle to follow another throughout the whole line.

221. If a single fibre of silk be wrapped round a piece of mirror glass, and another piece be laid on this with some pressure, the two pieces will adhere with a force sufficient to carry the lower piece. In this experiment, first performed by Huyghens, the surfaces are at the distance of the 2400th part of an inch. When the silk is not interposed, the force of adhesion is much greater. It is estimated to be 14 times

greater than gravitation. If the silk fibres are crossed in wrapping them on the glass, the distance is doubled, and there is no adhesion. The experiment succeeds in vacuo, and when there is no possible excitement of electricity. The adhesion must, therefore, depend upon a corpuscular attraction, operating at a small, but measurable distance.

When a polished plate of glass rests on another, in a horizontal position, the former presses on the latter, with a force equal to its weight. Yet, the plates are not in absolute contact: the distance between their surfaces, exceeds the distance between two coherent particles of a piece of glass, it is believed, many millions of times. Although they are not in contact, the lower plate supports the upper one, and prevents it from descending to a less distance: and the upper plate presses the lower one. These opposite and equal forces, are therefore repulsive. The glasses are said to be in physical contact, and the distance between their surfaces, is about the 4450th part of an inch.

The distance between two polished surfaces in physical contact, can be diminished by external force. It is found by experiment, that a pressure equal to 1000 pounds to a square inch is necessary to bring two polished surfaces of glass so near that light shall not be reflected at the interposed medium. Even under this intense pressure, the opposite particles of the two surfaces are not so near as to unite by the force of cohesion, but repel with a force equal to the external pressure. Two particles of glass, actually cohering, are supposed to be at a distance many thousand times less.

Now, it is well known, that the distance between cohering particles may be either *increased* or *diminished* without destroying their union. In the former case, they must be still within the sphere of their mutual cohesive attraction; and in the latter case, within the limit of a mutual repulsion. In either case, there is an inherent corpuscular force actually exerted, tending to restore the particles to their natural unrestrained distances. Hence, there is a repulsion when the distance is less than this; then, at an increased distance, there is





a cohesive attraction ; when the distance is farther increased, there is a repulsion ; and again, an attraction, when the last distance is doubled.

Boscovich conceives, that in the small and insensible distances in which physical contact is observed, and which do not exceed the 1500th part of an inch, there are many alternations of attraction and repulsion, according as the distance of the atoms is changed. Consequently, within this narrow limit, there are many situations in which two atoms neither attract nor repel. However this may be, the alternations stated above, are not hypothetical, but inferred from observation and experiment ; and may, therefore, be considered as constituting an experimental law of nature.

Experiments shew, that one piece of solid matter acts with a repulsive force, on another at a small distance, which is capable of being calculated ; and that this repulsive force is greater, as the distance is diminished. It appears too, that the repulsion between solids cannot be overcome by any force that we can employ ; that it is the immediate cause of the physical contact of bodies, and that, in all such cases, it is the force which we *commonly call pressure*.

#### PROP. 89.

222. The mutual action of two solid bodies in physical contact, is always in the direction, perpendicular to the common plane of contact.

*Fig. 49.* Let  $A B$  be the surface of a solid body, and let us suppose that the corpuscular force acts on a particle at all distances that do not exceed a certain line  $x$ . Let  $P$  be an external particle, whose distance from the surface  $A B$  is less than  $x$ , and  $D P$  a perpendicular to the surface. Let  $O C D E$  be a sphere, whose centre is in the particle  $P$ , and whose radius is equal to  $x$ .

It is plain, that a part  $C D E$  of this sphere is occupied by the matter of the body. Every particle, therefore, contained in the spherical segment  $C D E$ , acts on  $P$ , but no particle beyond the surface of the sphere can act on it. Now, suppose that the particle  $F$  repels  $P$  in the direction and with the force  $P H$ . There is another particle  $G$  similarly situated on the other side of the perpendicular  $P D$ , which repels  $P$  in the direction  $P I$  ; and the

measure of this force  $PI$  must be equal to  $PH$ , because the distance  $PG$  is equal to  $PF$ . The forces  $PH$  and  $PI$  compose a force  $PK$ , the direction of which must evidently be perpendicular to the surface  $AB$ .

Every particle  $F$  situated any where on one side of  $DP$ , will have a corresponding particle  $G$  similarly situated on the other side, provided the body be homogeneous; and all the forces which act on  $P$ , will have their resultant in the perpendicular direction  $PK$ .

Therefore, when two solids are in physical contact, every particle of each is repelled by all the particles of the other, whose distances do not exceed  $x$ , in a direction perpendicular to the common plane of contact.

If the solids attract, as when two polished plates of metal or glass adhere, the mutual force is exerted perpendicularly to the surface, as will appear from this demonstration, by substituting attraction for repulsion.

This proposition makes a great change in the method of considering mechanical problems; at least, it suggests a more natural one than that usually adopted, in all cases where the direction of a motion is affected by physical contact. Almost in every case in which the mechanician has been accustomed to resolve forces, nature takes the contrary method, and compounds them. Thus, in explaining the motion of a body descending along an inclined plane, the usual way is, to resolve the force of gravity into two others, one in the direction of the motion, the other perpendicular to that direction; the latter can neither accelerate nor retard the motion; therefore, the body moves, as if the former only affected it. But, the motion along the inclined plane is really a compound motion; and the moving force is the resultant of gravity, acting vertically, and of the repulsion of the plane, acting in the direction perpendicular to its surface, in every instant. The combination of these forces generates a force in the direction of the plane.





## II. OF EQUILIBRIUM IN GENERAL.

223. When all the parts of a solid body are at rest under the action of several forces, the body is said to be in equilibrium.

224. When a body is in equilibrium under the action of several forces applied to different points, the forces, also, are said to be in equilibrium with one another. This is not an accurate expression. One force cannot be in equilibrium with another, unless both act at the same point. If two equal forces be applied in opposite directions to the same point, all change of motion is prevented; and when two forces act without producing any change of motion, we infer that they are equal and opposite; they are in equilibrium. But, when several forces are *set in equilibrium* by the intervention of a solid mass, each force is immediately in equilibrium, not with another of the external forces, but with the corpuscular attraction or repulsion of the particle to which it is applied. For, the particle is supposed to be at rest; therefore, the corpuscular force is equal and opposite to the external force, (50).

### PROP. 90.

225. "When any number of forces are in equilibrium, by the intervention of a solid body, they are such as would be in equilibrium if they all acted on a single particle." Or,

When a solid body is in equilibrium under the action of several forces, these forces would be in equilibrium if they all acted on a single particle.

"One of the external forces A is in equilibrium, not with another of the external forces B, but with the corpuscular forces which connect the rest of the body with the point, or with the atom to which A is applied. In like manner, each of the other external forces is in immediate equilibrium, only with the force exerted by the point to which it is applied, which force is the corpuscular force connecting that particle with the rest of the body. Also, we may say in general, that the force exerted by any one of these particles is not a simple force, but the combined action of all the particles with which the particle acted on is *immediately con-*

nected. The force exerted by the particle acted on by the external force, is the equivalent of all those immediately connecting forces. Now, when forces are thus excited in distant parts, the excitement must take place over all the intervening particles, perhaps differently in each. We have supposed all the forces to be in equilibrio. Therefore, the equilibrium must obtain over all. For, if any particle be not in equilibrio, it will not remain in its present situation, contrary to our supposition of perfect rest over all. The equilibrium, therefore, is general. Now, consider how this equilibrium is produced. Every particle is attracting or repelling its adjoining particles, and is *equally* attracted or repelled by them. Therefore, the whole corpuscular forces are made up of pairs, and in each pair the two forces are equal and opposite.

It is evident, that if all these corpuscular forces were applied to one point, they would be in equilibrio, because each is opposed by its equal. But, when a number of forces, acting on one point, are in equilibrio, if they be divided into parcels, the equivalents of those parcels would be in equilibrio, if applied to one point. Now, in the present case, the forces exerted by the different points to which the external forces are applied, are each the equivalents of parcels of the corpuscular forces exerted all over the body. Therefore, these equivalents would be in equilibrio, if all were applied to one point. But each of those equivalents is equal and opposite to the external force with which it is in immediate equilibrio. Therefore, these external forces are such as would be in equilibrio, if all were applied to one point."

226. Hence, *first*, if a body be in equilibrio between two external forces, these forces are equal and opposite. They not only act in contrary directions, but the line which joins their points of application, coincides with the line of direction, in which the forces are exerted; for, by the preceding proposition, they would be in equilibrio if they acted at the same point, and by dynamics, their directions, when so acting, are opposite and lie in the same right line.

*Second.* When a body is in equilibrio between three forces, their directions lie in one plane. For, by dynamics this condition is necessary for their equilibrio, if applied to one point.

*Third.* When a body is in equilibrio between three forces, their intensities are proportional to the sides, and diagonal of a parallelogram, which lie in the directions of the forces, or to the sides of a triangle, which are *parallel* or *perpendicular*,





or equally inclined to those directions. For, the three forces would be in equilibrio, if they acted at one point, by the preceding proposition; and then each would be equal to the resultant of the other two, (55). Consequently, their intensities will have the proportion of those lines, (59, 60).

PROP. 91.

227. "If three forces are set in equilibrio, by being applied to three different points of a rigid body, all their directions meet in one point, or they are all parallel; and any one of the forces is to any other of them, reciprocally, as the perpendiculars drawn to their directions from the point to which the remaining third external force is applied."

*Fig. 50.* "Let the three forces, acting in the directions  $AD$ ,  $BE$ ,  $CF$ , be applied to the points  $A$ ,  $B$  and  $C$ , of a rigid body. Let their intensities be represented by the length of these lines. Then it is to be proved, that these directions either meet in one point  $T$ , or that they are parallel. Also, if  $CG$  and  $CH$  be perpendicular to  $AD$  and  $BE$ , we shall have  $AD : BE = CH : CG$ .

If the directions  $AD$  and  $BE$  are not parallel, let them meet in  $T$ . Join  $CT$ , and draw  $Ca$  parallel to  $AD$ , and  $Cb$  to  $BE$ .

Since we suppose these forces in equilibrio by the intervention of the rigid body, they are such as would balance, if applied to one point with the same directions and intensities. Now, the directions  $Ca$  and  $Cb$  are the same with  $AD$  and  $BE$ , and the figure  $CaTb$  is a parallelogram, and the forces  $AD$  and  $BE$  are in the same proportion as the sides  $Ca$  and  $Cb$ , and the third  $CF$  is as  $CT$ , and has that direction, because a force equal and opposite to  $CT$  would, if applied at  $C$ , balance the forces  $Ca$  and  $Cb$ . Therefore, the direction of the force applied at  $C$ , passes through the intersection  $T$  of the other two directions. Hence, the first part of the proposition is demonstrated.

Secondly, by reason of the parallels  $Ca$  and  $AT$ , and the parallels  $Cb$  and  $BT$ , the angles  $CbG$  and  $CaH$  are equal. So are the right angles at  $G$  and  $H$ . Therefore, the triangles  $CbG$  and  $CaH$  are similar, and  $Ca : Cb = CH : CG$ . But  $Ca : Cb = AD : BE$ . Therefore, we have  $AD : BE = CH : CG$ ."

If the directions of the two forces  $AD$  and  $BE$  do not meet, they are parallel. For, in this case the third force  $CF$  must be parallel to the other two, and equal to their sum, or to their difference, according as  $AD$  and  $BE$  act the same way or in contrary directions; because, if all acted at one point and in equilibrio,  $CF$  would equal the sum, or the difference of the other two.

### III. OF THE CENTRE OF GRAVITY.

228. The centre of gravity of a single body, is the centre of equal parallel forces, acting on all its particles. For, the force with which a mass gravitates, is the aggregate of all the elementary forces with which its equal integrant particles, gravitate. The place of the resultant of those elementary forces, retains the same invariable position in the mass, whatever may be the state of the mass, with respect to motion or rest, or whatever may be its position with respect to the direction of the force of gravity.

This is manifest from the principles of the action of parallel forces, (dynamics); and hence, the centre of gravity retains the same invariable position in the mass.

229. If a force equal to the weight of any body, be applied so as to pass vertically through its centre of gravity, and act in the opposite direction, the body will rest in any position, (78, 1st. 3d).

Hence, we may define the centre of gravity of a mass to be "that point, which, being supported, the body will remain "at rest in any position."

For, whether this remarkable point lie within the superficies of the solid, or be external to it, and unconnected with it, the contrary force which acts opposite to the resultant of the elementary forces of gravity, prevents motion in the mass, and consequently supports the centre of gravity.

230. In any system of bodies which gravitate with forces proportional to their masses respectively, their centre of gravity will coincide with the centre of parallel forces, which are proportional to the masses.

At the earth's surface, the force of gravity is proportional to the mass, since it is the measure of the mass; and in any system of bodies employed in our experiments, or arranged in machines, the gravitating forces which affect them severally, are considered as acting in parallel directions, on account of the smallness of the angle which the whole system subtends at the earth's centre.





231. If a force, equal to the whole weight of any system of bodies which are invariably connected with one another and with their common centre of gravity, be applied to this point, so as to act in the direction opposite to the resultant of their several gravitating forces, the system will be at rest in any position.

Hence, the centre of gravity of a system of bodies may be defined to be "that point situated between them, which, being "supported, the system will remain at rest."

232. The centre of gravity of a line, surface or solid, composed of homogeneous particles, may be found by determining the place of the resultant of equal parallel forces acting on all the particles.

In certain lines and figures, the centre of gravity evidently coincides with the mathematical centre.

*First.* The centre of gravity of a right line or row of equal particles, is the middle point of the line.

*Second.* The centre of gravity of the periphery of a circle, of an ellipse, of a parallelogram, of an equilateral triangle, of a regular pentagon, &c., is the centre of the figure. And the centres of gravity of the areas or figures themselves, are also the centres of the figures.

*Third.* The centre of gravity of a sphere, a regular ellipsoid, a cylinder or cylindroid whose ends are parallel, a regular tetrahedron, octahedron, &c. coincides with the mathematical centre.

Consequently, the centre of gravity may, in all these cases, be determined geometrically. It may likewise be found geometrically, in the cases stated in the following propositions.

#### PROP. 92.

233. To find the centre of gravity of a rectilinear triangle.

*Fig. 51.* Let  $ABC$  be any rectilinear triangle; and from the angles  $A$  and  $C$ , draw  $AD$ ,  $CE$ , bisecting the opposite sides  $BC$  and  $AB$ . The point of intersection  $G$ , will be the centre of gravity of the triangle.

Since  $\triangle A C D = \triangle A B D$ , each must contain the same number of homogeneous points or particles, all equally affected by parallel forces of gravity. And since every line that can be drawn in  $\triangle A B C$ , parallel to  $B C$  is bisected by  $A D$ , the equal elementary forces acting on every such line, have their resultant in  $A D$ . Therefore, the centre of all the elementary forces acting on  $\triangle A B C$ , must lie somewhere in  $A D$ . And, because the  $\triangle A C E = \triangle B C E$ , the centre of these forces must, by similar reasoning, lie in  $C E$ ; and consequently, it must be at  $G$ , the point of intersection of  $A D$  and  $C E$ . Therefore, this point is the centre of gravity of the triangle  $A B C$ .

PROP. 93.

234. The distance of the centre of gravity of a triangle, from the centre of gravity of the base, is to its distance from the vertex, as 1 to 2.

That is  $D G : A G :: 1 : 2$ , and  $D G : A D :: 1 : 3$ .

Since triangles of equal bases and of the same altitude are equal,  $\triangle A C D = \triangle A B D$ , and  $\triangle G B D = \triangle G C D$ . Therefore,  $\triangle A G C = A G B$ ; but  $\triangle A G B = 2 \triangle A G E$ . Therefore,  $\triangle A G C = 2 \triangle A G E$ . And since triangles of the same altitude are to each other as their bases,  $G C = 2 G E$ , and  $G E = \frac{1}{3} C E$ . (w.)

PROP. 94.

235. To find the centre of gravity of a trapezium.

*Fig. 52.* Let  $A B C D$  be a trapezium. Draw the diagonal  $A C$ , dividing it into two triangles  $A B C$ ,  $A D C$ , and find their centres of gravity  $E$  and  $F$ . Join these centres and divide  $E F$  in the inverse ratio of the triangles; that is, make  $G E : G F :: D P : B P$ . (The lines  $D P$  and  $B P$  being perpendiculars on the same base, are to each other as the triangles of which they are the altitudes).

$E$  and  $F$  are the centres of the elementary parallel forces with which all the particles of the two triangles gravitate respectively; or the places of the resultants. If, therefore,  $E$  and  $F$  express these resultants,  $E : F :: B P : D P :: G F : G E$ . Therefore, (by Prop. 23,) the point  $G$  is the place of the general resultant, or the centre of gravity of the trapezium  $A B C D$ .

Since any plane figure contained by a greater number of right lines may also be divided into triangles, the method of finding the centre of gravity of such figure is obvious.





## PROP. 95.

236. To find the centre of gravity of a triangular pyramid.

*Fig. 53.* Let  $A B C V$  be any triangular pyramid, whose base is  $A B C$ . Find  $D$  the centre of gravity of the base, and  $E$ , that of any side  $B C V$ . Join  $D V$  and  $A E$ , and the point of intersection  $G$  will be the centre of gravity of the pyramid.

Since  $D V$  passes through the centre of gravity of every plane in the pyramid parallel to  $A B C$ , the centre of gravity of the pyramid must lie somewhere in  $D V$ ; and for the like reason it must lie somewhere in  $A E$ . Therefore, it is at  $G$ .

## PROP. 96.

237. The distance of the centre of gravity of a triangular pyramid from the centre of gravity of the base, is to its distance from the vertex, as 1 to 3.

That is,  $G D : G V :: 1 : 3$ ; and  $G D : D V :: 1 : 4$ .

$E F = \frac{1}{2} V F$ , and  $D F = \frac{1}{2} A F$ ; therefore,  $E F : D F :: V F : A F$ . Therefore,  $\triangle D E F$  is similar to  $\triangle A V F$ ; and, being in the same plane,  $D E$  is parallel to  $A V$ , and it is equal to  $\frac{1}{2}$  of it. And because  $D E$ ,  $A V$  are parallel, the triangles  $D E G$ ,  $A V G$ , are similar. Hence,  $D E : A V :: G E : G A :: G D : G V$ . But,  $D E : A V :: 1 : 3 \therefore G E : G A :: G D : G V :: 1 : 3$ . Or  $G E = \frac{1}{3} G A = \frac{1}{3} A E$ ; and  $G D = \frac{1}{4} G V = \frac{1}{4} D V$ .

## PROP. 97.

238. To find the centre of gravity of a pyramid having any number of sides.

*Fig. 54.* Divide the base of the pyramid into triangles, and the pyramid into corresponding triangular ones. Find the centre of gravity of each triangular pyramid; and then, (by Prop. 23), find the centre of gravity of the whole.

Suppose the quadrangular pyramid  $A B D C V$ , to be divided into the two triangular ones  $A B C V$ , and  $B D C V$ , whose centres of gravity are  $H$  and  $F$ . Divide  $H F$  in  $G$ , making  $G F : G H :: A B C V : B D C V$ . The point  $G$  is the centre of gravity of the whole pyramid, (by parallel forces, Dynamics).

COR. 1. If  $A P$ ,  $D P$ , be perpendiculars on  $B C$ , the centre of gravity  $G$  may be found by this proportion:

$$A P + D P : A P :: H F : G F.$$

Because, the triangular pyramids are of the same altitude, they are to each other as their bases; and consequently, they are to each other as the perpendiculars  $AP$ ,  $DP$ . Hence, when  $H$  and  $F$ , the centres of gravity of the triangular pyramids, are determined;  $G$ , the centre of gravity of the whole pyramid, may be found by the above proportion.

**COR. 2.** Since any solid may be divided into pyramids, the centre of gravity, of a solid of any figure whatever, may be found by considering the centres of gravity of all the constituent pyramids as the points of application of parallel forces proportional to their masses respectively, and then finding the centre of these forces.

**PROP. 98.**

**239.** The distance of the centre of gravity of any pyramid from that of its base, is to its distance from the vertex, as 1 to 3.

Let  $K$  and  $E$  be the centres of gravity of the triangles  $ABC$ ,  $DBC$ , which compose the base  $ABDC$ ; and let  $I$ , be that of the whole base or trapezium  $ABDC$ , situated in the right line  $KE$ . Then since  $KH : KV :: EF : EV$ , the line  $FH$ , which joins the centres of gravity of the triangular pyramids, is parallel to  $KE$ .

Hence,  $GI : GV :: HK : HV :: FE : FV$ . But,  $HK : HV :: 1 : 3$ .

Therefore,  $GI : GV :: 1 : 3$ . Or,  $GI = \frac{1}{3} GV = \frac{1}{3} IV$ .

**COR.** Since the proposition can be demonstrated of a pyramid of any finite number of sides, it will evidently be true, when the number of sides is unlimited, and the pyramid becomes a cone. Hence, the distance of the centre of gravity of any cone from that of its base, is to its distance from the vertex, as 1 to 3.

**240.** The centre of gravity of any number of bodies whose masses are known, whether the bodies be invariably connected or not, may be found by the method (77,) of finding the place of the resultant of parallel forces. This is evident from article 230.

Although, the application of that method, to the finding of the centre of gravity of systems of bodies, may be sufficiently obvious to those who have made some proficiency, yet a few examples may be desired by the beginner. Besides, some useful corollaries, which would have burthened the general doctrine of forces, may be added with propriety, in this place.





## PROP. 99.

241. To find the centre of gravity of two bodies.

Find the centre of gravity of each, and divide the right line which joins these centres, in the inverse ratio of their weights or masses. The point so dividing this line, will be the common centre of gravity of the bodies.

*Fig. 55.* If we express the weights or masses of A and B, by A and B; then  $A + B : B :: A B : A G$ . The first three terms of this proportion being given, A G, and consequently the point G, may be found.

COR. 1.  $A \times A G = B \times B G$ . For  $A G : B G :: B : A$ , and the product of the extremes, = the product of the means.

COR. 2. If we take any point D in the right line A B D, the sum of the products of each mass into its distance from D = the sum of the masses into the distance of their common centre of gravity from that point.

Since  $A \times A G = B \times B G$ ;  $A \times D A - A \times D G = B \times D G - B \times D B$ .

Therefore, by transposing,  $A \times D A + B \times D B = \overline{A + B} \times D G$ .

## PROP. 100.

242. To find the common centre of gravity of any number of bodies situated in the same right line.

*Fig. 56.* Find the common centre of gravity of any two of them; then, suppose these two placed at that centre, and find the common centre of gravity of the aggregate and a third body. And so on for any number, (77).

Since the effect of the moving forces of A and B acting at A and B will be destroyed by the action of a force equal and opposite to their resultant, (71), just as if  $A + B$  were so applied as to act directly at the place of their resultant; it follows, that if G represents their sum, acting at their common centre of gravity, the common centre of gravity of G and E is that of A, B and E. That is, if  $E C : G C :: G (= A + B) : E$ , then the point C is the common centre of gravity of the three bodies A, B and E.

COR. 1. If F be the common centre of gravity of B and E; then  $(A + B) \times G C = E \times E C$ , also,  $(B + E) \times F C = A \times A C$ .

COR. 2. The sum of the products of the several masses into their distances from the common centre of gravity on one side = the sum of the similar products on the other side. That is  $A \times A C = B \times B C + E \times E C$ .

For, suppose  $F$  to be the centre of gravity of  $B$  and  $E$ , as  $G$  is that of  $A$  and  $B$ . Then, (by Cor. 2, Prop. 99). . . . .

$$B \times BC + E \times EC = \overline{B + E} \times FC.$$

But  $\overline{B + E} \times FC = A \times AC$ , (by Cor. 1, Prop. 100).

Therefore,  $A \times AC = B \times BC + E \times EC$ .

That this corollary is true of any number of bodies, may be seen, by conceiving  $A$  to be the centre of gravity of several masses  $H$  and  $I$ , whose aggregate  $= A$  which has been above considered as a single body. Let  $F = B + E$ .

Then as  $\overline{B + E} \times FC (= F \times FC) = A \times AC$ ;

$$\text{So } \overline{H + I} \times AC (= F \times FC) = \overline{B + E} \times FC.$$

It is evident that each body of the system may be considered as placed at the centre of gravity of several others, and then representing the sum of their masses.

COR. 3. Hence, the 2nd corollary, Prop. 99, is true of any number of bodies situated in a right line; and indicates the following ready method of finding the centre of gravity:

Multiply the weight of each body into its distance from the assumed point  $D$ ; then ascertain the sum of these products, and also, the sum of the weights; lastly, divide the former sum by the latter. The quotient will express the distance of  $C$ , the common centre of gravity from the point  $D$ .

Thus, let  $A = 30$  pounds,  $B = 10$  pounds, and  $E = 5$  pounds; and  $A$ 's distance from  $D = 18$  feet,  $B$ 's distance  $= 10$  feet, and  $E$ 's distance  $= 8$  feet.

Then  $30 \times 18 = 540$  the product of  $A$ 's mass and distance.

$10 \times 10 = 100$  the product of  $B$ 's mass and distance.

$8 \times 5 = 40$  the product of  $E$ 's mass and distance.

The sum of these products  $= 680$ , divided by 48, the sum of the masses, gives 14 feet 2 inches, which is the distance of  $C$ , the common centre of gravity, from the point  $D$ .

#### PROP. 101.

243. To find the centre of gravity of any system of bodies, situated in the same plane, or in different planes.

*Fig. 57.* Find the centre of gravity of any two  $A$  and  $B$ , (by Prop. 99); let this centre be  $E$ .

*Second.* Suppose  $E = A + B$ , that is, suppose the masses of  $A$  and  $B$ , to be concentrated at  $E$ ; and find  $F$ , the centre of gravity of  $E$  and a third body  $C$ .





*Third.* Suppose  $F = E + C = A + B + C$ ; and find  $G$ , the centre of gravity of  $F$  and  $D$ . Then  $G$  will be the common centre of gravity of the system  $A, B, C, D$ .

The propriety of this method is evident, from the principles of the composition of parallel forces, or from the reasons given in (242).

**PROP. 102.**

244. The sum of the products of each mass into its distance from any given plane is equal to the sum the masses into the distance of their common centre of gravity from the same plane.

From the centre of  $A$  draw  $A m$  parallel to any given plane  $O M$ , and from the centre of  $B$  draw  $B n$  parallel to  $A m$ . Then, if  $A$  be removed to  $a$  and  $B$  to  $b$ , points in the perpendicular drawn from their centre of gravity  $E$  to the plane  $O M$ , the distance of each from the plane will remain unchanged; and consequently, the products of their masses into their distances from  $O M$  will be the same. By similar triangles,  $E a : E A :: E b : E B$ . Therefore, the position of  $E$  their centre of gravity remains the same relatively to the bodies, and absolutely as respects its distance from the plane.

Now, if  $a = A$ , and  $b = B$ ;  $a \times a I + b \times b I = a + b \times E I$ , (by Cor. 2. prop. 99). That is,  $A \times A N + B \times B L = E \times E I$ , when  $E = A + B$ .

By applying the same method to the second and third steps, employed in the preceding proposition, it can be shewn that  $A \times A N + B \times B L + C \times C M + D \times D O = A + B + C + D \times G H$ .

**PROP. 103.**

245. If a system of bodies, be invariably connected with any right line passing through its centre of gravity, and the gravitating forces of the bodies, be reduced to this line in perpendicular directions, the centre of gravity of the system, will retain its former position in the same line.

*Fig. 58.* Let  $A, B, C, D$ , be a system of bodies invariably connected with the right line  $E H$ , passing through  $G$  the centre of gravity.

*First.* Let  $A, B, C, D$ , be in the same plane in which  $E H$  lies, and suppose  $m$  to be the centre of gravity of  $A$  and  $B$ ; draw  $A E$ ,

B F, perpendicular to E H, and suppose these bodies to act at E and F; then  $m$  is reduced in the perpendicular direction  $m K$ , and  $E K : E K :: A m : B m$ , (by Prop. 17). Also, let  $n$  be the centre of gravity of C and D; when these bodies are in like manner transferred to H and I, their centre  $n$  will be reduced to L, and  $H L : I L :: C n : D n$ .

Now, because A and B may be reduced to  $m$ , and C and D to  $n$ , without disturbing G, (by demonstration of 242), it follows, that when  $m$  lies on one side of E H, then  $n$  must lie on the other side; otherwise, G would not be in the right line which connects  $m$  and  $n$ ; but G must lie in  $m n$ , (by parallel forces). And since  $K G : L G :: m G : n G$ , the centre of gravity of the system (G) retains the same position in E H, when the gravitating forces of A, B, C, D, are reduced in perpendicular directions to this line; that is, when they are made to act at the points, E, F, I, H.

*Again.* Since A, B, C, D, may each be considered as the centre of gravity of masses or particles situated in other planes, it is evident, that the centre of gravity of any system remains undisturbed by reducing the gravity of its parts, to any right line, passing through the centre of gravity of the system.

*Note 1.* A rule for determining the centre of gravity of curve lines, of curve surfaces, and of most solids bounded by curves, requires for its investigation an application of the higher geometry.

*Note 2.* The *centre of inertia* is called the *centre of gravity* by the writers on Physics, with a few distinguished exceptions. Dr. Robison calls this point the *centre of position*. Dr. Thomas Young, and others, prefer the term *centre of inertia*. In considering the equilibrium or other conditions of gravitating forces at the earth's surface, the term centre of gravity is both proper and significant. Yet, as a general expression for that point, which is in all cases supposed to retain its position unchanged in a mass or system, it is not so eligible: for, the centre of gravitating forces of a system of connected bodies is not the same point, when the system is supposed to be removed near to the centre of the earth, or to a distance very remote from its surface; and the common centre of the gravitating forces of the earth and moon is continually shifting its position or relative distance from these bodies. The properties ascribed to the centre of gravity of celestial masses or systems are strictly the properties of the centre of inertia, and this centre has properties independent of gravity.





### III. OF THE EQUILIBRIUM OF GRAVITATING FORCES.

246. It is a general law of the equilibrium of gravitating forces, that the system will be at rest, when the point to which the support is applied coincides with the centre of gravity.

It is also a general law of equilibrium, that when the point to which the support is applied is in the vertical line that passes through the centre of gravity, the system will be at rest. Or more strictly, if a system be brought to rest, with the centre of gravity, and the point to which the support is applied, in the same vertical line, it will remain at rest.

These laws are easily deduced from the principles of the action of opposite parallel forces (Dynamics); and are found to obtain in all operations in which mechanical equilibrium are concerned. It is evident, that when the upward force (or support) acts in the direction of the vertical line passing through the centre of gravity, it is opposed, directly, or by the intervention of equal atomic forces, to the resultant of all the elementary forces with which the particles of the system gravitate.

That vertical line in which the point of support is situated, is called, by way of distinction, *the vertical line*.

#### PROP. 104.

247. When any accidental force disturbs the equilibrium of the system, the centre of gravity quits the vertical line: and the supporting and gravitating forces generate a resultant which prevents quiescence.

*Fig. 59.* Let the system A B C be suspended as the fixed point S, about which it is supposed to be capable of moving freely. If brought to rest, with G its centre of gravity in the vertical line V S, it will remain at rest, and all the forces will be in equilibrio. But, if impelled to any other position *a b c*, its centre of gravity being now at *g*, it cannot remain quiescent when the disturbing force ceases to act. The resultant of the elementary forces of gravity Q now acts at *g*, and is greater than P the pressure on the support. The action of the support  $p = P$  the pressure. The forces Q and  $p$  act in contrary directions, and the place of their

resultant  $R$  is at some point  $h$ , on the other side of  $Q$  the greater. This resultant is a moving force, acting the same way as  $Q$ .

In the erect position of the figure, the force  $R$  will cause  $g$  to approach  $G$ ; in the inverted position, it will cause  $g$  to recede from  $G$ .

248. The tendency of a system to return again to its quiescent position, or the tendency of the centre of gravity, to return to the vertical line, constitutes the stability of the equilibrium.

The quiescent position of the system, and the character of the stability, are determined by the position of the centre of gravity with respect to the point of support. And since there are various degrees of stability of equilibrium, it may, like other variable quantities, be either positive, or negative, or evanescent.

#### PROP. 105.

249. If the centre of gravity, be situated below the fixed point, at which any system is freely supported, the equilibrium will be positively more or less stable.

*Fig. 59.* When the system is erect, as in the figure, and the centre of gravity  $G$  is situated below  $S$ ; any force which disturbs the equilibrium will cause  $G$  to ascend. Suppose it has ascended to  $g$ . The moment it has quitted the vertical line  $VP$ , in which  $S$  is situated, a resultant is generated which is a moving force, acting on the otherside of  $Q$ , and acting the same way. When the disturbing force has ceased to act, this resultant  $R$ , will cause the centre of gravity to descend again from  $g$  to  $G$ .

250. Although  $R$  diminishes as the centre of gravity in its descent approaches  $G$ , and vanishes when they coincide, yet a momentum has been acquired which will cause the centre of gravity to pass the vertical line  $VS$ , and the system will oscillate for some time before it comes to rest. As often as the centre of gravity passes the vertical line on either side, it ascends.

Hence the consequential law with respect to the stability of equilibrium, *that when the centre of gravity must ascend if*





*displaced from the vertical line, the equilibrium is of the stable kind.*

**PROP. 106.**

**251.** When the centre of gravity is situated above the fixed point, at which any system is freely supported, the stability will be negative.

*Fig. 59.* By inverting the figure, it may represent any system, whose centre of gravity  $G$ , is above the point of support  $S$ . When the centre of gravity quits  $VP$ , it descends; and the instant it is displaced, a resultant is generated, which tends to carry it farther from  $G$ . When, therefore, the disturbing cause is withdrawn, this force will prevent the system from regaining its former quiescent position. The stability of the equilibrium, when  $G$  is above  $S$ , is therefore negative.

**252.** If there be no obstacle to free rotation about  $S$ , the centre of gravity, having quitted the position  $G$ , will descend in a semicircle to the vertical line on the other side, where it finds a position of positive stability, and after several oscillations, it will come to rest.

The conditions of this case of negative stability, indicate the consequential law of equilibrium; *that when the centre of gravity must descend, upon quitting the vertical line, the stability of the equilibrium is negative.*

**253.** When the centre of gravity, coincides with the point of support, the equilibrium is evanescent or neutral. The system will remain at rest in any position, and will change it for another, from the slightest disturbing causes. Since the centre of gravity is always in the vertical line, no resultant moving force is generated, by varying the position.

**254.** If an oval body of uniform density rest on the extremity of its minor axis, upon a plane, its equilibrium will be stable.

For, if its quiescence be disturbed, the centre of gravity ascends, and advances more slowly than the vertical line. It as-

cends, because the semi-diameters joining it and the successive points of contact, increase in length. And, because it falls behind the vertical line, a resultant is generated, tending to restore it to its former position.

**255.** If an oval body rest on the extremity of its major axis, upon a plane, its equilibrium will be tottering, or its stability negative.

The semi-diameters, joining the centre of gravity and the point of contact, diminish as that point recedes from the vertex; the centre of gravity must, therefore, approach the plane when it quits the vertical line; that is, it descends; and a resultant is generated, tending to remove it farther from its former position.

**256.** If a body, having a plane base, rest on a sphere, and the centre of gravity be a less distance from the point of contact than the radius of the sphere, the equilibrium will be stable. If the centre of gravity be a greater distance from the point of contact than the radius of the sphere, the equilibrium will be tottering. And at the distance of the radius, the stability is evanescent.

In the first case, the centre of gravity ascends when disturbed: in the second case, it descends: and in the last case, its initial motion is horizontal.

**257.** A cylinder may rest on an inclined plane, and have positive stability, if its centre of gravity be at a less distance from the line of contact, than its geometrical axis. But, in this case, the centre of gravity does not lie in the vertical plane passing through the line of contact: it is situated towards the higher part of the plane; so, that if the cylinder be impelled down the plane, the centre of gravity ascends initially. If, when the centre of gravity is brought near to the vertical plane, (yet still situated towards the upper part of the inclined plane,) it be abandoned to the actions of gravity and of the inclined plane, it will again descend; and the cylinder will roll up the plane, during the relative descent of its centre of gravity.





258. The stability of a sphere resting on a horizontal plane is neutral, rather than evanescent. It rests indifferently on any point of its surface; and when disturbed, its centre of gravity describes a horizontal line, and never quits the vertical line. No resultant moving force can, therefore, be generated by merely changing the point of contact. It advances in consequence of the momentum communicated to it; and is finally brought to rest by the resistance of the air, and the friction on the plane.

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#### IV. OF THE MECHANICAL POWERS.

259. The simplest arrangements of solid matter, by which we are aided in moving heavy bodies, or in overcoming any resistance, are called mechanical powers; and by a combination of these, all compound machines are formed.

At least six simple arrangements of solid matter, each differing from the others very distinctly in its general plan, may be enumerated as mechanical powers. 1. *The lever.* 2. *The wheel and axle.* 3. *The pulley.* 4. *The inclined plane.* 5. *The wedge.* 6. *The screw.* Some add, *the rope, or funicular machine.*

260. In the mechanical powers and in all their combinations, the force to be overcome, or the body to be moved or kept in equilibrium, is called *the weight*, and the force required for either of these purposes, is called *the power*.

#### THE LEVER.

261. The lever is an inflexible bar, moveable about a line fixed in position, which is its axis of motion. The force which supports the axis of motion in an invariable position is called the prop or fulcrum.

262. The use of the lever, is, to raise some weight, or to overcome some resistance, acting at one point, by means of a weight or other force, applied to another point. The points in the lever, at which these forces would remain quiescent, are the points of equilibrium.

263. Levers are distinguished into three kinds, according to the relative positions of the fulcrum, weight, and power.

A lever is said to be of the *first order*, when the fulcrum, or axis of motion, is between the weight and power.

A lever is of the *second order*, when the weight is between the fulcrum and the power.

A lever is of the *third order*, when the power is between the fulcrum and the weight.

PROP. 107.

264. Supposing any straight lever to be without gravity, the weight and power, acting in parallel directions, will be in equilibrio, when they are inversely as the distances of their directions from the fulcrum.

If we consider the lever as an inflexible line, the proposition has been already demonstrated in all its cases, in the theorems on parallel forces, (Dynamics). If we consider it as a rigid mass devoid of weight, the proposition is demonstrated in (227). For the perpendiculars on the directions, drawn from the fulcrum, measure the distances, and these perpendiculars are inversely as the weight and power.

*Fig. 60.* In a lever of the first order,  $P = W$ , or  $P > W$ , or  $P < W$ ; and  $Q = P + W$ ; and  $P : W :: A C : B C$ .

*Fig. 61.* In a lever of the second order,  $P < W$ ; and  $Q = W - P$ ; and  $P : W :: A C : B C$ .

*Fig. 62.* In a lever of the third order  $P > W$ ; and  $Q = P - W$ ; and  $P : W :: A C : B C$ .

COR. 1. In every lever, of the three forces, the *fulcrum*, *power* and *weight*, that which *countervails* or *opposes* the other two, acts between the other two; is equal to the sum of the other two; and has its direction in the same plane.

COR. 2. When the weight and power are given, the ratio of the distances of their points of equilibrium is given; since  $P : W = A C : B C$ , always, when there is an equilibrium.





COR. 3. When in any lever, the weight and power are in equilibrium,  $P \times BC = W \times AC$ .

COR. 4. In any straight lever, if the power be removed to a greater distance from the fulcrum, then, in order to restore the equilibrium by a change in the weight, this change must be proportional to the change of the distance of the power.

Since  $W \times AC = P \times BC$ , and since  $AC$  and  $P$  are constant multipliers,  $W$  must vary as  $BC$  varies.

COR. 5. The moving force\* of the power, when the weight remains unchanged, and the power is removed to a greater distance from the fulcrum, is proportional to the difference of the products into the distances. That is, if  $P$  be removed from  $B$  to  $E$ , the moving force of  $P = P \times EC - W \times AC$ ; or moving force of  $P = P \times EC - P \times BC = P \times BE$ .

COR. 6. The moving force of the power is proportional to its distance from the point of equilibrium. It is as  $P \times BE$ ; and  $P$  being a constant multiplier, it is as  $BE$ . Or, since the moving force may be measured by the increment of the weight necessary to prevent motion, and since the increment of the weight necessary to this effect, is proportional to the increment of the distance (Cor. 4), the moving force of  $P$ , is proportional to  $BE$ , the distance of the point of application, from that of equilibrium.

#### PROP. 108.

265. Supposing any bent or crooked lever to be without gravity, or supposing the power and weight to act at an angle, there will be an equilibrium when the power and weight are inversely as the perpendiculars on their directions, drawn from the fulcrum.

This proposition has been demonstrated in (227).

Figs. 63, 64.  $P : W :: aC : bC$ ; and  $P \times bC = W \times aC$ .

COR. 1. The power and weight are inversely as the sines of the angles which their directions make with the arms of the lever; for, these lines are the perpendiculars on the directions.

COR. 2. The effective force of the weight or power : is to its whole force :: as the perpendicular on the direction : to the length of the arm at which it acts.

From the point of application of  $P$ , erect the perpendicular  $BD$ , and describe about it the parallelogram  $bE$ . This will be the parallelogram of forces; for, each triangle  $BED$ ,  $BbD$ , has

\* By the moving force of the power, is to be understood, its efficacy in displacing the weight, or in moving the lever from a state of rest; and not the energy of the power, depending on an existing motion.

its sides perpendicular to the directions in which they act; and the whole force of  $P$  : resistance of the lever : their resultant : :  $b B (= D E) : b D (= B E) : B D$ ; or by similar triangles : :  $C B : b B : C b$ . That is, the effective resultant : is to  $P$  : :  $C b$ , (the perpendicular on the direction of  $P$  :  $C B$  (the length of the arm at which it acts.)

**COR. 3.** When in any lever the weight and power are in equilibrium, and the lever is moved by a foreign force, the momentum of the power will be equal to that of the weight. For, the extremities of the arms of a straight lever, or of the perpendiculars, (when the lever is bent or the forces oblique,) will describe similar arcs in the same time, and the arcs described are as the radii: therefore, the radii are as the co-temporaneous velocities. Consequently, the momentum of  $P$  : momentum of  $W$  : :  $P \times b C : W \times a C$ , and these are equal.

**PROP. 109.**

266. In a physical lever of any form or materials, there will be an equilibrium, when the product of the power into the perpendicular, drawn from the *vertical line* to its direction + the product of the weight of the arm at which it acts, into the horizontal distance of the centre of gravity of the arm at which it acts = the sum of the like products, on the side at which the weight acts.

*Fig. 65.* That is,  $P \times b c + M \times m c = W \times a c + N \times n c$ ; supposing  $M$ ,  $N$ , to be the centres of gravity of the arms, and these to represent the weights of the arms respectively.

If the lever were devoid of weight, it would be  $P \times b c = W \times a c$ . If the power and weight were removed, the gravitating forces of the arms would be as  $M \times m c$  and  $N \times n c$ , which may be equal or unequal; but, when there is an equilibrium, the point  $c$  is in the direction of the resultant of all the parallel forces of gravity (229). Therefore,  $P \times b c + M \times m c = W \times a c + N \times n c$  (Cor. 2, Prop. 100), when there is an equilibrium.

**COR. 1.** If any number of weights are distributed on one arm of the lever, and several powers act on the other, there will be an equilibrium when the sum of the products of all the  $P$ 's into their distances = the sum of the products of all the  $W$ 's into their distances.

**COR. 2.** The sum of the momenta of the powers = the sum of the momenta of the weights, (Cor. 3, Prop. 108).

267. In any lever, balanced on a fulcrum, the equilibrium will be stable when the centre of gravity is below the point at





which the fulcrum acts ; that is, below the centre of libration. When the centre of gravity is situated above this point, the equilibrium will be tottering. When the centre of gravity and the centre of libration coincide, the equilibrium will be neutral.

In the first case, the centre of gravity, when it quits the vertical line, ascends ; in the second case, it descends ; and in the last case, it continues in the vertical line in every position of the lever, and therefore, the lever will rest indifferently in any position.

### THE WHEEL AND AXLE.

268. The *wheel and axle* consists of two cylinders of unequal radii, moveable about the same axis, and with which they are invariably connected. The greater cylinder, is called the *wheel*, and the less the *axle*.

The power is supposed to act in the direction of a tangent at the circumference of the wheel, and the weight in the direction of a tangent to a circle of the cylinder, on the other side of the vertical plane. Also, the wheel and axle, are supposed to be of uniform density, so that the centre of gravity of the machine, shall always remain in the axis of motion, and consequently in the *vertical plane*.

In a single wheel and axle, the power and weight are applied by means of ropes, and as these have sensible thickness, half the thickness of the rope is to be added to the radius to which it is applied.

#### PROP. 110.

269. In the wheel and axle, there will be an equilibrium, when the power is to the weight, as the radius of the axle, is to the radius of the wheel.

For, when the perpendicular distances of the directions of the power and weight from the vertical plane, are inversely as these forces, the centre of forces, or centre of gravity of P and W, will lie in the vertical plane, and be supported. Therefore, the power and weight will be in equilibrio, when

$$P : W :: R_{\text{axle}} : R_{\text{wheel}}.$$

If the power and weight act obliquely to the radius, the power will be to the weight inversely, as the perpendiculars on their directions, drawn from the centres of the wheel and the axle. And as these perpendiculars are a maximum, when the power is applied in a tangential direction, this direction is the most advantageous in which it can be applied.

## THE PULLEY.

In the following propositions, the weight of the pulley is not taken into consideration.

270. When the weight or resistance acts in the direction of gravity, the weight of the pulley being added to it, will give the total resistance. Otherwise, the total resistance will be as the diagonal of a parallelogram, whose sides have the proportion and directions of the resistance and the gravitating force of the pulley.

### PROP. 111.

271. In a simple pulley, there will be an equilibrium, when each of the two powers is to the weight, as the radius of the pulley is to the chord of the arc which is embraced by the rope.

*Fig. 66.* Let the powers  $P, P$ , acting by the ropes  $MC, NE$ , passing over the circles  $M, N$ , and round the pulley  $CGE$ , whose centre is  $D$ , be in equilibrio with the weight  $W$  suspended from the pulley.

Since the forces  $P$  and  $P$  are equal, in consequence of their being in equilibrio, they may be represented by the equal lines  $WA, WB$ ; then, drawing  $AF$  parallel to  $MW$ , and  $BF$  parallel to  $NW$ , the diagonal  $WF$  will represent the resultant of the two powers  $P, P$ . But, since this resultant is in equilibrio with the weight  $W$ , acting in the opposite direction  $FW$ , we shall have  $FW$  for the measure of the weight  $W$ .

If we join  $CD, ED$ , and draw  $CE$ , we shall have the two isosceles triangles  $WBF, CDE$  similar; because the angles  $A$  and  $D$ , comprehended between the equal sides, are both the supplement of the same angle  $CWE$ , the quadrilateral figure  $CDEW$  having its angles  $C$  and  $E$  right angles.

Hence,  $WB : BF : WF = DC : DE : CE$ ; therefore,  $P : W$ , or  $P : W$ , as radius to the chord of the arc embraced by the rope.





**COR. 1.** Each of the powers is to the weight, as radius is to twice the cosine of the angle, which each rope makes with the direction of the weight. As,  $\angle DCH = \angle CWD$ ,  $CE = 2$  cosine, or twice  $CH$ .

**COR. 2.** When the two ropes are parallel, there will be an equilibrium if each of the powers  $P, P$ , is one half of the resistance  $W$ ; for in this case, the arc embraced by the rope, is  $180^\circ$ , the chord of which is equal to twice the radius, and therefore each of the powers = one half the weight or resistance.

**COR. 3.** The powers act with the greatest advantage, when the ropes are parallel: for in this case, the chord of the arc embraced by the rope, is a maximum.

**COR. 4.** When one of the ropes is attached to a fixed support, then, to maintain the weight in equilibrio, only one power is required. As the fixed support sustains what was otherwise sustained by the second power, the single power will keep the weight in equilibrio, when  $P = \frac{1}{2} W$ , and the ropes parallel.

**272.** If  $n$  be any number of moveable pulleys connected with a fixed support by the ropes on one side, and with each other by the ropes on the other side, there will be an equilibrium, when  $P : W :: 1 : 2n$ , and the ropes parallel.

**273.** In a system of fixed and moveable pulleys, that is, in a *muffle*, there will be an equilibrium when the power is to the weight, as unity to the number of ropes which draw the moveable pulleys, the ropes being parallel.  $P : W :: 1 : \text{number of ropes}$ . And there is the same proportion, when the ropes are in different planes.

### THE INCLINED PLANE.

**274.** *The inclined plane* is any plane surface which makes an angle with the horizon. *The inclination of the plane* is the angle which its upper surface makes with the horizon. In considering the action of this mechanical power, three surfaces are supposed to be so arranged, as to form the sides of a triangular prism, whose perpendicular section is a right angled triangle; the hypotenuse lies in the inclined plane, one of the sides is horizontal, and the other vertical. These

lines, are called the *length*, *base*, and *height* of the inclined plane, respectively.

The inclined plane is generally employed to keep a heavy body in equilibrium, by means of a force which has not a vertical direction; or to enable a force, less than its weight, to raise it to some height.

PROP. 112.

275. In the inclined plane there will be an equilibrium, when the power is to the weight, as the height of the plane is to its length, the direction of the power being *parallel to the plane*.

That is,  $P : W :: AC : AB$ .

*Fig. 67.* Let  $abD$ , be a section of the body or weight  $W$ , and  $ABC$ , a section of the inclined plane, by a vertical plane passing through  $(G)$  the centre of gravity of the body.

In order that an equilibrium may exist, the resultant of  $P$  and  $W$  must lie in the perpendicular to the plane which passes through their point of concurrence; for, the plane acts perpendicularly to its surface, and in that case only, can its action counter-vail their resultant and prevent motion.  $GD$  perpendicular to  $AB$  is, therefore, in the direction of the resultant of the weight and power. Let the sides of the parallelogram  $FW$ , be in the directions in which the power and weight act, that is,  $FG$  parallel to the plane, and  $GW$  a vertical line, and therefore parallel to  $AC$ . Then  $\triangle s GWD$  and  $ABC$ , are similar; and  $DW = FG : GW :: AC : AB$ . But,  $P : W :: DW : GW$ .

Therefore,  $P : W :: AC : AB$ .

COR. The weight : the power : pressure on the plane :: radius : sine : cosine of the plane's inclination.

*Note.*—When a body is placed upon a horizontal plane, it cannot be in equilibrio, unless the vertical line, passing through its centre of gravity, also passes through a point in the touching surface or base. When a body is placed on an inclined plane, the same condition is necessary to its equilibrium; and besides, another force  $P$ , is required to prevent its sliding down the plane. When the base is broad, friction may serve this purpose.





## PROP. 113.

276. In an inclined plane there will be an equilibrium, when the power acting *parallel to the base*, is to the weight, as the height of the plane is to its base.

*Fig. 67.* Let  $f w$  be a parallelogram, of which the diagonal  $g d$  is perpendicular to the plane  $A B$ , the side  $g w$  perpendicular to the base  $B C$ , and  $g f$  perpendicular to the vertical end  $A C$ . When there is an equilibrium, the perpendicular action of the plane will be equal and opposite to the resultant of  $P$  and  $W$ ; hence,  $g d$  will be the direction of that resultant, and  $f w$  the parallelogram of forces.

Therefore,  $P : W :: g f : g w$ ; and by similar triangles,  $P : W :: A C : B C$ .

COR. The pressure on the plane : power : weight :: radius : sine : cosine of the plane's inclination.

## THE WEDGE.

277. A *wedge* is a triangular prism of wood or metal, one edge of which is introduced into an opening in any body, for the purpose of separating its parts, or of fixing them at a certain distance.

Besides the triangular pieces employed for cleaving wood, *piles*, *nails*, and *pins*, employed by architects; *coulters*, *spades*, and other utensils, used in agriculture; *swords*, *knives*, *chisels*, and other cutting instruments, are varieties of wedges.

The power applied to the wedge, is commonly of the nature of an impulse, as the blow of a ram or hammer; but sometimes it is a pressure.

The wedge is either *isosceles* or *scalene*, according as the section, by a plane perpendicular to its sides, is isosceles or scalene. The surface on which the power acts, is called the *back* or *head* of the wedge; those on which the resistances or weights act, are called the *sides* of the wedge; and the head and sides are represented by the triangular section perpendicular to the sides of the prism.

## PROP. 114.

278. In a wedge which is not wholly immersed in the cleft, there will be an equilibrium, when the power, acting perpen-

dicularly on the head of the wedge, is to the resistances on the sides, as the head of the wedge is to the sum of the sides.

*Fig. 68.* The action of the body  $MN$ , on the sides of the wedge  $ABC$ , is in directions perpendicular to the sides,  $AC$ ,  $BC$  (222). And the action on the head  $AB$  is, by hypothesis, perpendicular to  $AB$ . Let  $WD$ ,  $WE$ , be the perpendicular directions of the forces on the sides, and  $PF$  of that on the back. Since there is an equilibrium, the three forces  $P$ ,  $W$ ,  $W$ , would be in equilibrio if they all acted at one point (225). Therefore, the sides of the triangle  $ABC$ , perpendicular to the directions of the three forces, are proportional to them respectively: that is  $AB : AC : BC :: P : W : W$  (226).

And  $P : W + W :: AB : AC + BC$ .

**COR. 1.** The force on the head is to the resistance on one side, as the head is to that side. Or, since halves have the same ratio to each other as the wholes, the power is to either resistance, as half the head is to the side on which that resistance is exerted. And this holds true, whether the wedge be isosceles or scalene.

#### PROP. 115.

279. In a wedge which is wholly immersed in the cleft, there will be an equilibrium, when the power acting perpendicularly on the head, is to the resistances on the sides, as the head of the wedge is to the sum of the sides of the cleft; the sides being measured from the head of the wedge, to the extremity of the cleft.

*Fig. 68.* Let the wedge  $abc$ , be wholly immersed in the cleft  $DGE$ . Now, the action of the wedge, at  $a$  and  $b$  on the body is perpendicular to the sides of the cleft  $DG$ ,  $EG$ . And as the action of the power is perpendicular to  $ab$ ,  $G$   $ab$ , is the triangle of forces; and  $P : w :: ab : aG : bG$ . And  $P : w + w :: ab : aG + bG$ .

#### THE SCREW.

280. The screw consists of a right cylinder of wood or metal, grooved on the surface in such a manner, that the projecting ridge forms a spiral plane, always equally inclined to a plane, perpendicular to the axis of the cylinder. The spiral plane going once round the cylinder, is called the thread





of the screw. A nut or hollow cylinder has a groove on its concave surface, corresponding to the thread of the screw; and when a circular motion is given, either to the screw or the nut, the other being fixed, the nut has a progressive motion in the direction of the axis of the cylinder.

While the nut performs one circular revolution, it describes, by its progressive motion, a space equal to the distance between two contiguous threads, or it advances through the height of one thread.

The power is applied by means of a lever; and the axis of the screw is the axis of motion.

PROP. 116.

281. In the screw there will be an equilibrium when the power is to the weight, as the height of one thread is to the circumference of a circle, whose radius is the distance of the point at which the power acts from the axis of the screw.

Let B C D represent a section of the screw by a plane perpendicular to its axis, and C E a part of the spiral thread upon which the weight is sustained. Then, C E is a portion of an inclined plane, whose height is the distance between two threads, and base equal to the circumference B C D. Let P = the power (acting at C, in the plane B C D, and in the direction perpendicular to the radius A C) which sustains the weight W.

When there is an equilibrium  $P : W :: \text{height} : \text{base}$  (276) :: height of one thread : circumference B C D.

Now, if, instead of P acting at C, we suppose  $p$  to act at G, on the lever G C A whose centre of motion is A, and to act in the same plane and direction, and there to preserve the equilibrium; and let the weight act at one point C, on the thread or inclined plane C E. Then  $p : P :: C A : G A$  (264) :: circ. B C D : circ. F G H. And we have these two proportions:

$$P : W :: \text{height of the thread} : \text{circ. B C D}$$

$$p : P :: B C D : \text{circ. F G H}$$

$$\text{By comp. } p : W :: \text{height of the thread} : \text{circ. F G H.}$$

If the weight be supposed to be diffused over the whole thread, then the power at G which can sustain any part of it : is to that part :: as the height of one thread : to the circ. F G H. And the sum of all these powers =  $P : \text{sum of all the corresponding weights} = W :: \text{height of one thread} : \text{circ. F G H.}$

This is in substance the demonstration given by Mr. Wood in his *Mechanics*, and is chosen for its simplicity. A more perfect one may be seen in the article *Mechanics*, new Edin. Encyclopædia.

**COR.** The power of the screw increases, as the distance between two threads diminishes.

**PROP. 117.**

**282.** When there is an equilibrium in any of the mechanical powers, and it is put in motion by any force, the velocity of the power : is to the velocity of the weight :: as the weight : is to the power.

*Case 1.* In the *lever*, the cotemporaneous velocities, are as the indefinitely small arcs, described by the points of application of the power and weight, with radii joining those points, and the centre of motion, when the lever is straight and the directions of *P* and *W* parallel; and generally, the cotemporaneous velocities are as the indefinitely small arcs, described by the extremities of the perpendiculars *a c*, *b c*, (fig. 63, 64,) in the same instant. Since the arcs are as the radii or perpendiculars *a c*, *b c*, these are as the velocities; and  $P : W :: a C : b C$ ;  $\therefore P : W :: V, W : V, P$ .

*Case 2.* In the *wheel and axle*, while the power moves uniformly through a space equal to any arc of the circumference of the wheel, the weight will move uniformly through a space equal to a similar arc of the axle; because the wheel and axle are invariably connected with the same axis of motion, and have constant radii. The spaces described in the same time, are as the velocities, and therefore, the velocities are as the arcs; or as the radii with which they are described, or as the perpendicular distances of the points of application from the vertical plane. But, these distances are inversely as the power and weight; therefore,  $P : W :: V, W : V, P$ .

*Case 3.* In a single moveable pulley, when the ropes are parallel, and that on one side is attached to a fixed point so that one power only is employed, and there is an equilibrium; then, if the weight be raised uniformly through any space in a given time, each of the ropes must be so much shortened, and the power must descend uniformly through twice that space in the same time.

Hence,  $P : W :: V, W : V, P$ ; and the same can be shewn of every system of pulleys.

*Case 4.* In the inclined plane  $P : W :: A C : A B$ , and the vertical descent of *P* : vertical ascent of *W* ::  $A B : A C$ .





Hence, when the velocities are reduced to the vertical direction,  $P : W :: V, W : V, P$ .

*Case 5.* In the wedge,  $P : W : W :: A B : A C : B C$ ; and the velocity with which the point F descends : velocities with which D and E recede :  $A C + B C : A B$ . That is,  $P : W + W :: V, W + V, W : V, P$ .

*Case 6.* In the screw, while the power moves uniformly through the circumference F G H, the weight moves uniformly through the height of one thread; and  $P : W :: \text{height of thread} : \text{circ. F G H}$ . Therefore,  $P : W :: V, W : V, P :: \text{height of thread} : \text{circ. F G H}$ .

**COR.** Hence, the well known principle in Mechanics, that, in all combinations of the mechanical powers, whatever is gained in power, is lost in time.



## V. OF MOTIONS AND PRESSURES ON INCLINED PLANES AND SOLID CURVES.

283. Under this title are considered those motions and pressures, which depend on the uniformly accelerating force of gravity, and the action of the surface at which they take place. The combination of these generates a moving force in a direction deviating from that of gravity, and less than it. Their combination also generates a quiescent pressure acting both ways at the surface, as two equal and opposite forces must do. (32, 222).

### PROP. 118.

284. The motion of a body on an inclined plane is uniformly accelerated.

*Fig. 70.* Let A B be an inclined plane, and G a body placed on it. Let D F be a parallelogram, the side G D being in the direction of gravity, and the side G F perpendicular to the plane. Then, G D and G F will be to each other, as the weight of the body to the corpuscular repulsion of the plane, which acts perpendicularly to its surface (222); and G H parallel to the plane is in the direction and proportion of the resultant of G D and G F.

Since  $GD$  is constant,  $GF$  is constant; and consequently  $GH$  is constant. But  $GH$  is the accelerating force. Therefore, the acceleration on an inclined plane is uniform.

**COR.** The whole force of gravity, the perpendicular resistance of the plane or the pressure on its surface, and the accelerating force, are to each other, as the length, the base, and the height.

By similar triangles  $GD : HD (= GF) : GH :: AB : CB : AC$ .

**PROP. 119.**

**285.** The space which a body would describe in falling freely : is to the space it would describe in the same time on the plane :: as the length of the plane : to its height.

For, the spaces which different uniformly accelerating forces cause to be described in the same time, are as the forces;  $a = s$ , (100). But  $A : a :: AB : AC$ , (Prop. 118, Cor. 1). Therefore,  $S : s :: AB : AC$ .

**COR. 1.** The velocity acquired by falling freely : is to the velocity acquired on the plane in the same time :: as its length : to its height.

For, when the time is the same,  $v = s$ , (111). But  $S : s :: AB : AC$ , by the proposition. Therefore,  $V : v :: AB : AC$ .

**COR. 2.** If, from the angle  $C$ , a perpendicular  $CE$  be drawn to the plane  $AB$ : then  $AC$  will be described by a body falling freely, in the same time that  $AE$  will be described by a body on the plane. For, by similar triangles  $AB : AC :: AC : AE$ .

**COR. 3.** If on the vertical line  $AC$  as a diameter, a circle be described, the times of descending along any chords  $AE$ ,  $EC$ , are equal, and also equal to the time of falling through the diameter  $AC$ .

Complete the parallelogram  $ACDE$ . The angle at  $E$  in the semi-circle is a right angle, and  $CE$  is perpendicular to  $AE$ . Therefore,  $AE$  and  $AC$  will be described in the same time (Cor. 2.) And  $CD$  is perpendicular to  $EC$ . Therefore,  $EC$  and  $ED$  will be described in the same time. But  $ED = AC$ ; consequently,  $AE$ ,  $EC$ , and  $AC$ , will all be described in the same time.

**PROP. 120.**

**286.** The time of descending along the plane : is to the time of falling freely through its height :: as its length : is to its height.





That is,  $T$  the time in  $AB : t$  the time in  $AC :: AB : AC$ .

The time in  $AC =$  time in  $AE \therefore T^2 : t^2 :: AB : AE$  (106).  
The triangles  $ABC$ ,  $ACE$ , are similar (Euclid 8, 6,); and  
 $AB$ ,  $AC$ ,  $AE$ , are continual proportionals.

Therefore,  $AB : AE :: AB^2 : AC^2$ . Therefore ex æquali

$$T^2 : t^2 :: AB^2 : AC^2.$$

$$\text{Or } T : t :: AB : AC.$$

Cor. The times of descending along different planes of equal altitude are to one another as the lengths of the planes.

PROP. 121.

287. The velocity acquired by descending along an inclined plane is equal to that acquired by falling through its height.

In uniform motions  $s = vt$ ; and in uniformly accelerated motions, the spaces described from the beginning = half the spaces described with the final velocity continued unvaried, or

$s = \frac{vt}{2}$ ; and since 2 is constant,  $s = vt$  is the latter case also, where  $v$  is the acquired velocity.

Therefore,  $AB : AC :: VT : vt$ .

And,  $AB : AC :: T : t$  (by Prop. 120).

Therefore,  $VT : vt :: T : t$ . Hence,  $V = v$ .

To make the ratio  $VT : vt =$  the ratio  $T : t$ ,  $V = v$  of necessity.

Cor. 1. The velocities acquired by descending along planes of different lengths, but of equal altitude, are equal.

Cor. 2. The velocities acquired by descending along any planes are to each other as the square roots of the altitudes of the planes. For, the velocities acquired by descending freely through their altitudes, are as the square roots of the altitudes.

PROP. 122.

288. The velocities acquired, by descending along any chords of a vertical circle which meet at the lower extremity of its vertical diameter, are proportional to the lengths of the chords.

*Fig. 71.* Let  $AB$  be the vertical diameter, and  $CB$ ,  $DB$ , chords meeting at its lower extremity. Also, let  $V$  = the velocity acquired in descending along  $CB$ , and  $v$  = the velocity acquired in descending along  $DB$ .

Now,  $CB$  and  $DB$  are described in the same time (Prop. 119, Cor. 3). But, when  $t$  is the same,  $v = s$ , (111). Therefore,  $V : v :: CB : DB$ ; and the acquired velocities are as the chords.

**PROP. 123.**

289. The times of descent along planes equally inclined to the horizon, are as the square roots of their lengths.

*Fig. 72.* The accelerating force is evidently the same on both planes. By the theorem  $s = t^2$ , (106),

$$T^2 : t^2 :: AB : DE, \text{ and } T : t :: \sqrt{AB} : \sqrt{DE}.$$

**PROP. 124.**

290. The velocity acquired, by descending along any number of contiguous planes, is equal to that acquired by falling freely through their whole height; supposing no diminution to be produced by the shock received at the angles.

*Fig. 73.* Let  $AD$ ,  $DE$ ,  $EB$ , be contiguous planes, and  $AF$ ,  $FG$ ,  $GC$ , their altitudes, whose sum is  $AC$ . By the proposition, the velocity acquired in  $AD$  = the velocity acquired in  $AF$ , and the velocity acquired in  $DE$  = that in  $FG$ . Therefore, the velocity at  $E$  = the velocity at  $G$ . For like reasons, the velocity at  $B$  = velocity at  $C$ .

**PROP. 125.**

291. If a body descend along a curve surface, its final velocity will be the same, as if it had fallen freely through the height of the curve; abstracting the effect of friction.

*Fig. 74.* The velocity at  $C$  is the same, whether the body may have descended perpendicularly through  $DC$ , or from the same height through  $AC$ . Let  $DC = \frac{1}{2} SB$ , the radius of the circle  $EBC$ . Then, if no velocity is lost when the descending body enters the curve at  $C$ , none will be lost in the arc  $CB$ . For, if gravity should cease to act, when the body is just entered on the curve at  $C$ , it would proceed uniformly with a velocity, always equal to that acquired by falling through  $DC$ , (161, Cor.)





But if mere change of direction in the arc of a circle, *whatever may be its curvature*, destroys no part of the acquired velocity, it is evident that mere change of direction in any continuous curve does not.

Or thus :

*Fig. 75.* Let  $A B, B C$ , be contiguous planes, making a finite angle at  $B$ , and suppose a body to have descended from  $A$  to  $B$ , and to have acquired a velocity which would carry it over  $B F$  in a given time, as  $1''$ . From  $F$  draw  $F D$ , perpendicular to  $B C$ , and with the radius  $B F$ , describe the arc  $F I$ . Draw  $B E$  parallel and equal to  $F D$ ; and  $B D$ , the diagonal of the parallelogram  $E F$ , is the space which will be described on the plane  $B C$ , in the given time,  $1''$ . Therefore,  $I D (= B F - B D)$ , is the velocity lost by the shock at  $B$ .

When an arc is diminished without limit, its versed sine is diminished without limit, with respect to the arc or its chord; because the diameter : is to the chord : : as the chord : is to the versed sine. But,  $I D$  is the versed sine of  $\angle F B C$ . If, therefore, this angle be infinitely small, as in a continued curve,  $I D$  will be infinitely small with respect to the arc of that infinitely small angle; so that  $I D$  becomes an infinitely small quantity of the second order. Therefore the velocity lost, is an infinitely small quantity of the second order. Therefore, although the repetitions of this loss were continual on a curve surface, it cannot, in any finite space, amount to a finite quantity. Therefore, the final velocity will be the same, as if the body had fallen freely through the perpendicular height of the curve.

**COR.** A body which has descended in any curve to the lowest point, from any height, will ascend to the same height, in same curve, continued upwards, or in any other curve.

#### PROP. 126.

292. The *spaces* described by two bodies, acted on by varying forces, whose ratio, in similar instants of time or points of space, is that of equality, are as the squares of the times from the beginning, or as the squares of the last acquired velocities.

*Fig. 76.* Let  $A G D, a g d$ , be similar polygons, similarly divided into rectangles and triangles; so that,  $A E : A F :: a e : a f$ , and  $A E : A G :: a e : a g$ . Then, if  $A G$  and  $a g$  be proportional to the times, in which two spaces are described by motions continually accelerated from rest;  $a, e, f, g, h$ , will be instants in the less time, similar to  $A, E, F, H, G$ , in the greater. And if  $A E = E F = F H = H G$ , and also if  $E B, F C$ , &c. be proportional to the velocities in the instants  $E, G$ , &c.; and also,

$e, b, f, c$ , &c. proportional to the velocities in the similar instants,  $e, g$ , &c.; and again, if the forces be supposed to act uniformly in the intervals between the similar instants, changing only at the instants  $E, F$ , &c. and  $e, f$ , &c.; then,

The areas  $AEB, EC, \dots$  will be as the spaces described in the equal successive intervals of the greater time; and the areas  $ae b, ec, \dots$  will be as the spaces described in the corresponding intervals of the less time. For, the rectangle  $E I$  is as the space described with the velocity  $EB$ , in the interval  $EF$  (110); and  $\triangle BIC$  is as the space described by the acceleration, in the same interval. Therefore, the trapezium  $EC$  is proportional to the space described in the interval  $EF$ . It may, in like manner, be made evident, that  $FL$ , is proportional to the space described in the interval  $FH$ ; and so of all the rest. The same is evidently true of all the trapeziums corresponding with the similar intervals of time in  $ag$ . Hence, the area  $AGD : agd :: S$  in time  $AG : s$  in time  $ag$ .

Now, because  $EC : FL :: ec : fl$ , similar portions of the two spaces are described by the two motions, in similar parts of the times. Hence the motions are similarly varied in similar points of the spaces described, or in similar instants of the times of description.

By similar figures  $\triangle ABE : abe :: AE^2 : ae^2 :: EB^2 : eb^2$ ; and the area  $ACF : acf :: AF^2 : af^2 :: FC^2 : fc^2$ ; and  $AGD : agd :: AG^2 : ag^2 :: DG^2 : dg^2$ . Let the similar parts of  $AD$  and  $ad$  be diminished in length, and augmented in number, to infinity. Then  $AD$  and  $ad$  will be similar curves; and still  $AGD : agd :: AG^2 : ag^2 :: DG^2 : dg^2$ .

Hence, when two forces act with continually varying intensities, causing the accelerations to vary incessantly; if the ratio of the forces in similar instants of the times, or similar points of the spaces, is that of equality; then the spaces described from the beginning will be to each other as the squares of the times elapsed, or as the squares of the last acquired velocities.

*Note.* In this demonstration, the velocities are supposed to vary by increments in the acceleration; for the triangular areas, proportional to the spaces described by the accelerations, in successive equal intervals of either of the times, continually augment; and hence,  $AD, ad$  are convex inwards. But, if these lines were concave inwards, the triangular areas corresponding to equal parts of  $AG, ag$ , would continually diminish, and the velocities would vary by decrements of acceleration. It is plain that the same demonstration applies to this case also.

The theorems relating to acceleration, in the first part, admit of various and most useful extensions; the principal of which are,

First,  $a \times t = v$ ; and farther,  $V : v :: A \times T : a \times t$ ; when  $A$  and  $a$  are accelerating forces of similar action,





Second,  $a \times \dot{s} = v \dot{v}$ ; and farther,  $V^2 : v^2 :: A \times S : a \times s$ .

Third,  $a \times (t^2) = s$ ; and farther  $T^2 : t^2 :: \frac{S}{A} : \frac{s}{a}$ .

These theorems are true of all similar actions; but the demonstrations, although not difficult, are too operose for an elementary synopsis.

#### PROP. 127.

293. The times of descent along similar arcs of similar curves, their chords being equally inclined to the horizon, are as the square roots of their lengths.

It is indifferent whether the curves be convex or concave upwards.

*Fig. 77.* Let  $A B$  and  $a b$  be similar arcs whose chords are equally inclined to the horizon. Then the accelerating forces are similar and equal; because, at similar points in the two curves the perpendicular actions of the surfaces, combined with gravity, give equal resultants in both. Therefore,  $T$  in  $A B : t$  in  $a b :: \sqrt{A B} : \sqrt{a b}$ , (Prop. 126).

COR. 1. The times of descent are as the square roots of the axis of the curves. For the axis of similar curves are to each other as the lengths of the similar parts.  $T$  in  $A B : t$  in  $a b :: \sqrt{B C} : \sqrt{b c}$ .

COR. 2. The acquired velocities are as the square roots of the arcs, (Prop. 126); and, therefore, as the square roots of the axis.

### VI. OF THE MOTIONS OF BODIES OR SYSTEMS ABOUT FIXED AND MOVING CENTRES.

294. A body, freely suspended from a fixed point or line, especially when employed in measuring time, is called a pendulum. In considering the motion of a pendulum, the whole of its matter is supposed to act at its centre of oscillation, or to be collected in that point.

When a pendulum is drawn aside, from the vertical line which passes through the point of suspension, and then abandoned, it descends again. When it is removed from the vertical line or axis of rest, it must ascend in an arc of a circle. Thus removed, the corpuscular force of the suspending line acts at an angle with the force of gravity. The combination of these forces generates a resultant in the direction of the tangent of the arc, which urges the pendulum towards the axis of rest. This resultant is an accelerating force, continually diminishing during the descent through the arc ; but the velocity, acquired at the lowest point, carries the body through an equal arc on the other side of the axis of rest. The resultant vanishes in this axis ; on the other side, it becomes negative ; or it acts as a retarding force, during the ascent of the pendulum. When the pendulum has ascended to the same height from which it descended, it is brought to rest. During its ascent, the retarding force had been continually augmenting ; and now it becomes again a positive moving force, accelerating the motion in the retrograde direction, during the descent to the axis of rest.

295. The motion of the pendulum, from the highest point on one side of the axis of rest, to the highest point on the other, is called an oscillation or vibration.

#### PROP. 128.

296. When a pendulum is proceeding from rest to motion, at any distance from the axis of rest, the whole force of gravity : is to the tension of the string : is to their resultant : : as radius : cosine : sine, of the angle contained by the direction of the string and the axis of rest.

*Fig. 78.* Let B C be the axis of rest, and A the momentary position of the pendulum. Draw A G in the direction of gravity, and E G parallel to A C, meeting the tangent E A in E. Draw A D, perpendicular to B C. A G E is the triangle of forces ; and  $A G : E G : E A :: \text{gravity} : \text{tension} : \text{their resultant}$  by similar triangles : :  $A C : D C : A D :: \text{radius} : \text{cosine} : \text{sine}$  of  $\angle A C B$ .





**COR. 1.** The force which accelerates or retards the motion of the pendulum, varies as the sine of the angle A C B.

For the direction of gravity is always vertical, and the force of tension is always perpendicular to the curve or its tangent. Therefore, E A will vary as the *accelerating force varies*, uninfluenced by the conditions of momentum. Consequently, this force will vary as A D varies.

297. Since the action of the suspending line, like the action of a solid surface, is always perpendicular to the momentary direction of the motion; the force which accelerates a pendulum in any curve, is exactly equal to that which accelerates motion on a solid surface of similar curvature, and similarly situated with respect to the horizon.

**PROP. 129.**

298. The times, in which pendulums of unequal length describe similar arcs of circles, are as the square roots of the lengths of the pendulums.

*Fig. 77.* For, in similar points of the similar arcs, the accelerating forces are in the ratio of equality. Therefore, the times will be as the square roots of the arcs (Prop. 126); and consequently, as the square roots of the radii, with which they are described. That is,

$$T, A B : t, a b :: \sqrt{A B} : \sqrt{a b} :: \sqrt{B C} : \sqrt{b c}.$$

$$\text{And } T, A B + B D : t, a b + b d :: \sqrt{B C} : \sqrt{b c}.$$

**PROP. 130.**

299. The velocity of a pendulum in its lowest point is proportional to the chord of the arc it has described in its descent.

*Fig. 79.* For  $v$ , chord  $A B = v$ ,  $A D = v$ , arc  $A B$  (Prop. 121, 125, and 297.)

And  $v$ , chord  $a b = v$ ,  $a d = v$ , arc  $a b$ . Therefore, the velocity acquired, in descending either arc, is equal to that acquired in descending its chord. But the velocities acquired by descending along the chords  $A B$ ,  $a b$ , are as the chords, (Prop. 122); and consequently, the velocities acquired, in descending along the arcs, are as their chords.

## PROP. 131.

300. The time of a single oscillation of a pendulum, supposing it to descend in the chord of the descending arc, and to ascend in that of the ascending arc, is equal to the time in which a heavy body would fall through eight times the length of the pendulum.

The time of descent along the chord  $AB$  = time of falling through  $2CB$ , (prop. 119, cor. 3); and the velocity acquired at  $B$ , will cause the pendulum to ascend the equal chord  $BE$  on the other side, in an equal time. But the falling body having described a space =  $2CB$  during the time of descending  $AB$ , will describe a space =  $2CB \times 3$  in the succeeding equal time of ascending  $BE$  (108.) Therefore, while  $ABE$  is described on the chords, a heavy body would fall through four times the diameter of the circle of which  $BC$  is radius, or through eight times the length of the pendulum.

COR. The motion in the arc is swifter than in the chords. If the pendulum oscillating in circular arcs beat seconds, it must be 39, 11 inches in length. Now, if this pendulum be supposed to move in the chords of the descending and ascending arcs of vibration, however small the arcs may be taken, a heavy body would fall through 26, 73 feet in the time of such oscillation; for  $39, 11 \times 8 = 312, 88$  inches = 26, 73 feet, nearly.

But a heavy body falls through about 16, 1 feet, during a single oscillation of the same pendulum, moving in the circular arc  $ABE$ , that is, in a single second: hence, the motion in the arc is much swifter than in the chords.

## PROP. 132.

301. If pendulums of unequal lengths revolve in horizontal circles, the suspending lines describing cones of equal altitude, their times of revolution will be equal.

Fig. 80. Let  $ABC, abc$ , be two cones thus generated, and  $CL$  their common altitude. The sides of the triangle  $ADG$  are in the directions of gravity, tension of the string, and centrifugal force: hence,  $AG : AD : DG :: \text{gravity} : \text{tension} : \text{centrifugal force}$ .

First, suppose  $A = a$ . The triangles  $ADG$  and  $ACL$  are similar; and gravity  $A : \text{its centrif. force} :: CL : AL$ . So





likewise, gravity  $a$  : its centrif. force ::  $C L : a L$ . And since  $A = a$ , centrifugal force  $A$  : centrifugal force  $a$  ::  $A L : a L$ . That is, the centrifugal forces, are as the radii of the circles which the moving bodies describe. Therefore the times of revolution are equal (prop. 60, corol).'

*Second*, suppose  $A$  and  $a$  to be unequal. Then all the forces affecting them, will be proportional to their masses. Hence, their velocities will be in the same ratio as before; and under the conditions of the proposition, the times of revolution will be equal.

*Cor.* If several bodies revolve in the concave surface of a parabolic conoid, describing horizontal circles of different diameters, round a vertical axis, their revolutions will be performed in equal times. For, the normals are equal to the equivalent strings; and the subnormals are the altitudes of the cones described by the former, and are equal.

302. If a flexible line be applied to any curve, and by a motion of one of its extremities, it be gradually unbent, another curve will be described, called the evolute of the first, and that curve, the involute of the second.

303. The radius of curvature of the evolute, is the unbent portion of the flexible line, with which any elementary arc of the evolute is described.

As the line unbends, the radius of curvature increases; and consequently the curvature of the evolute diminishes.

The radius of curvature of any elementary arc of the evolute, is evidently equal to the arc of the involute from which it was evolved.

#### PROP. 133.

304. The radius of curvature is always perpendicular to the evolute.

If the involute were a polygon, the evolute would be composed of circular arcs, to each of which the describing radius would be perpendicular. This must be so, however numerous the sides of the polygon. Consequently, when the number is infinite, or the involute becomes a continued curve, the radius of curvature of any point in the evolute, will be perpendicular to this curve, or to its tangent at the same point.

305. The radius of curvature of any point of the evolute, is a tangent at some point of the involute. And the point of contact is the centre of curvature of the element of the evolute described with it as radius. Also, this element of the evolute has the same degree of curvature, as a circle described with the unbent portion of the line as radius.

PROP. 134.

306. If any plane figure roll on another, and any point in it describe a curve, that curve will always be perpendicular to the right line joining the describing point and the point of contact.

Suppose the figures rectilinear polygons. Then, while the rolling figure turns on one angle, the describing point will pass through an arc of a circle, and while it turns on another angle, that point will pass through another arc. Consequently, the figure described will be composed of circular arcs meeting each other at finite angles. Each constituent arc will be perpendicular to the radius vector belonging to it, although the radii of no two arcs meet in one point. Now, if the number of sides of the figures be increased without limit, the polygons will approach infinitely near to curves, and each elementary arc, passed through by the describing point, will still be perpendicular to the radius vector.

307. If a circle roll along a right line, the curve described by a point in the circumference, while it moves from one place of contact to another, is called a cycloid. The curve thus described is the common cycloid.

A point within the circumference of the rolling circle, describes a curve, called by some, a trochoid, by others, a prolate or inflected cycloid.

A point without the circumference, describes a curtate or looped cycloid.

308. The rolling circle, with which the point describing any cycloid is connected, is called the generating circle.





The base of the cycloid is the straight line that joins the extremities of the curve, or that joins the points in which that, which traces the curve, begins and ends its motion.

The right line which bisects the base at right angles, and terminates in the curve, is called the axis ; and the point in which it meets the curve, is the vertex.

309. The base of any cycloid is equal to the circumference of the generating circle ; and the segment of the base passed over, is equal to the arc intercepted by the first touching point, and that which terminates the segment.

310. The axis of any cycloid is equal to twice the right line that joins the describing point and the centre of the circle. The axis of the common cycloid is equal to the diameter of the generating circle.

#### PROP. 135.

311. The evolute of a cycloid is an equal cycloid ; and the length of the arc of the involute is double that of the tangent, cut off by the vertical tangent.

*Fig. 81.* Let two equal circles  $A B$ ,  $B C$ , rolling on the parallel lines  $D A$  and  $E B$ , at the distance of a diameter of the circles, describe with the points  $F$  and  $G$ , the cycloids  $E F$ ,  $E G$ . The circles being equal, the cycloids will be equal.

The curve  $E G$  is the evolute of  $E F$ . Draw the diameter  $F H$ . Then  $H$  will be the point that coincided with  $D$ , and  $H A = D A = E B = \text{arc } B G$ , and the remainders  $A F$ ,  $G C$ , are equal. Therefore,  $\angle A B F = C B G$ , (for each is equal to half the angle subtended by the same arc at the centre, Eu. 20, 3,) and  $F B G$  is a right line, (Eu. 15, 1). But  $F G$  is perpendicular to  $A F$ , because  $A F B$  is an angle in a semicircle. Therefore,  $F G$  touches the curve at  $F$ , and it is always perpendicular to  $E G$ , (prop. 134). Therefore,  $F G$  is the radius of curvature of the element  $G$ , of the curve  $E G$ . And as this is so at every instant from the beginning of the motion, the evolute of  $E F$  will coincide with the equal cycloid  $E G$ ; that is, the evolute of a cycloid is an equal cycloid.

And the arc  $E F$  is always equal to  $F G$  (303), or to  $2 F B$ , or to double the portion of the tangent at  $F$ , cut off by the vertical tangent  $E B$ .

**COR.** The semi-cycloid  $EV = 2TV$ , or twice the diameter  $BC$  of the generating circle; and the whole cycloid  $EM = 4BC$ .

**PROP. 136.**

312. The fluxion of the cycloidal arc, is to that of the base, as the evolved radius to the diameter of the generating circle.

*Fig. 82.* For the increment  $GI = 2BK$ ; and  $BK : BL :: BG : BC$ , and  $2BN : BL :: FG : BC$ , which is therefore the ratio of the fluxions.

**COR. 1.** Since  $BC$ , the diameter of the generating circle, is constant,  $BL$  is constant; and since  $FG$ , the radius vector, or its half  $BG$ , is continually increasing until it comes to the vertex,  $GI$  is continually increasing. If the base be described by a uniform rotation of the generating circle, the first half of the cycloid will be described with an accelerated, and the second, with a retarded motion.

**COR. 2.** If the fluxion of the base be constant, that of the curve will vary as  $BG$ , the distance of the describing point from the point of contact varies. By the above proportion  $GI (= 2BK) = FG$ ; and halves of quantities vary as the wholes. Therefore,  $GI = BG$ .

**COR. 3.** The velocity at the vertex is twice as great as that at the base. For, when  $I$  is at the vertex,  $BK$  coincides with  $BL$ ; and then  $GI = 2BL$ .

313. A cycloidal pendulum is a pendulum suspended by a flexible line between two cycloidal cheeks; so that the suspending line applies alternately to the two cheeks, and the pendulum describes an equal cycloid by the evolution of the line, (135).

**PROP. 137.**

314. A cycloidal pendulum descends in large and small arcs to the lowest point in equal times.

*Fig. 83.* The accelerating force in the direction of the curve or its tangent, is always to the force of gravity as  $GC : BC$ ; and





since  $BC$  is constant, the accelerating force varies as  $GC$ , or as  $GV$ , which is double of  $GC$  (135). That is, the accelerating force is every where as the arc to be described.

Suppose two arcs terminated at  $V$  to be equally divided, and the number of parts in one, equal the number in the other; and let the number of parts in each be so great that the spaces shall be evanescent; then, the force will be every where as the small space to be described. The force may be considered for each evanescent space as uniform: and the increments of time in which all the evanescent spaces composing one arc, are described, will be equal to the increments of time in which those of the other arc are described; and consequently, the whole times will be equal.

**PROP. 138.**

**315.** The time in which a cycloidal pendulum performs one oscillation : is to the time in which it would fall through half the length of the suspending line : : as the circumference of a circle : is to its diameter.

Supposing the generating circle to move uniformly, the velocity of the describing point  $G$  will always be as  $GB$  (312); or (since  $BP : GB :: GB : BC$ , and  $GB = \sqrt{BP \times BC}$ ) as  $\sqrt{BP}$ .

But, the velocity of a body falling in  $BP$ , or descending in  $EG$ , varies in the same ratio (106, 113, Dynamics, and 291 of this division). Therefore, if the velocity at  $V$  = that in falling through  $TV$ , the point  $G$  will always coincide with the place of a heavy body descending in  $EGV$ . And the velocity of the point of contact  $B$  is half that of  $G$ , at  $V$ , (312);  $B$  will, therefore, describe (by its uniform motion,) a space equal  $TV$  in the time of falling through  $TV$ , (107), and will describe  $ET$  in a time : which is to that time :: as  $ET : TV$ , or as half the circumference of a circle to its diameter. And the time of describing  $ET$  will be the time of descent in the cycloidal arc, at half the time of one oscillation, since  $G$  arrives at  $V$  when the point of contact is at  $T$ .

Therefore, the time of a whole oscillation : time of falling through  $TV$  ( $= \frac{1}{2} SV$ ) :: as the circumference of a circle : to its diameter.

**PROP. 139.**

**316.** The times of oscillation of different cycloidal pendulums are as the square roots of their lengths.

For, the times of falling through half their lengths are as the square roots of these halves, or of the wholes.

**PROP. 140.**

317. The time of oscillation of a pendulum moving in circular arcs, is the same as that of a cycloidal pendulum of equal length, when the circular arcs are indefinitely small; but in larger arcs the times are greater.

For, in small cycloidal arcs, the radius of curvature is nearly constant; but, at greater distances from the vertex, the circular arc falls without the cycloidal, and is less inclined to the horizon.

**COR. 1.** The oscillations of a pendulum moving in very small circular arcs are nearly isochronous; and perfectly so, when the difference between the greatest and least oscillations is evanescent.

**COR. 2.** The time of an oscillation in a small circular arc is the time of falling through half the length of the pendulum, as the circumference of a circle to its diameter.

**PROP. 141.**

318. The space which a falling body will describe during one oscillation in a small circular arc, or in a cycloid: is to half the length of the pendulum :: as the square of the circumference of a circle: is to the square of its diameter.

When the accelerating force is uniform  $s = t^2$ ; and the time of an oscillation: time of descent through half the length of the pendulum :: circumference: diameter. Therefore, the space fallen through during one oscillation: half the length of the pendulum :: square of the circumference: square of its diameter.

**PROP. 142.**

319. A falling body will describe 16 1-12 feet nearly, while a pendulum, the length of which is 39,2 inches, performs one of its least oscillations.





The circumference of a circle : its diameter :: 3 . 1415 : 1.  
 And the space described by the falling body :  $39 \div 2$  ::  $3,1415^2$  :  $1^2$ ; by the preceding proposition. Consequently, the space described =  $19,6 \times 9,86902225$  ( $= 3,1415^2$ ) = 193 inches, or 16 1-12 feet, nearly.

**COR. 1.** To find the space through which a heavy body will fall during one oscillation of a pendulum of any length, multiply half its length by 9,869, and the product will be the space required, expressed in the terms which measure the pendulum.

**320.** If a direct force be applied to any perfectly smooth figure, resting on a perfectly level plane altogether free from asperities, in a line parallel to the plane, the figure will slide along the plane. There will be no rotation. For, the force applied, acting equally on every particle, by the medium of the atomic forces of the impelled body, and acting in a direction parallel to the plane, which, therefore, gives no resistance, all the particles will move with equal velocities. On the supposition that the motion is not resisted in any way by the plane, a sphere, so impelled, would not roll, but slide along it.

**321.** But, if the touching surface be a curve, the least diminution of the velocity of the touching point will give rise to rotatory motion. The impulse applied, and the resistance causing the reduction of the velocity of the touching point, are unequal parallel forces acting in contrary direction at different points. It would obviously require a force effectively equal to the resistance, acting the same way on the other side of the centre of inertia, to restore equality of velocity to all the particles; that is, to prevent rotation.

Since there is no surface absolutely without asperities or resisting points, a body with a curve surface resting on a plane, cannot be moved along it by any direct force without generating rotation.

**322.** It is sufficiently obvious, that the rotation which takes place in confined motions, and which depends on the contact

of other matter, is really produced by an opposing force acting at the point of contact. And, as this force is not compensated by any consequential action elsewhere, in the direction of the progressive motion, it follows, that in all such cases of rotation, there is a diminution of the quantity of progressive motion which would otherwise result from the impulse.

323. The motion of any particle of a body rolling on a plane, is variously modified by the curvature of its surface. Thus the centre of inertia of a homogeneous sphere describes a right line, and the axis of rotation describes a parallel plane, with an uniform motion; while all other points describe curves less contracted, as they are nearer the centre, and with varying velocities: but no point in a cylindroid, rolling on a plane, describes a right line; and the curves which its particles describe differ from those described by the particles of the sphere.

324. When a point describes a conic section, or any curve except a circle, there cannot be a quiescent centre of motion. No single point within the curve can remain at rest, with respect to the moving point, for any finite time. But, the centre of curvature to any point in the curve, (that is, the centre of the equicurve circle) may be considered as a quiescent centre of motion in the instant the moving point is passing through the corresponding point in the curve. The radius of curvature is supposed not to vary during the moment in which an element of the curve is described. And accordingly the centre of curvature is called the *momentary centre of motion*.

In every succeeding instant, some adjacent point becomes the momentary centre of motion; so that in such cases the motion may be said to be about a moving centre. This is well illustrated by the cycloidal pendulum. Perhaps the elementary arc at the vertex of the cycloid described by the pendulum, is the only part of the curve that can, with strict propriety, be said to have a quiescent centre, even for a moment.





## PROP. 143.

325. The momentary centre of motion of a point describing the element of a cycloid, at the vertex, is in the perpendicular to this element, at the distance of twice the diameter of the generating circle.

*Fig. 83.* For the point S, whose distance from the vertex = 2 B C, is the centre of the equicurve circle with which this element coincides. Therefore, this point is the momentary centre of motion of the point V. (210, 324).

Or thus:

Since the fluxion at V : fluxion at T :: V S : V T, the fluxion at S = 0; for V T will be one side of an elementary triangle, of which the increment at V is the base, and S the vertex.

Hence, vel. at V : vel. at T : vel. at S :: 2 : 1 : 0, and S is for an instant at rest.

326. That point in a body oscillating about a fixed axis, in which, if the whole of its matter were collected, the time of oscillation would remain the same, is called the *centre of oscillation*. Or, the *centre of oscillation* is that point in the axis of the pendulum, of which the distance from the axis of motion measures its effective length, with respect to the time of oscillation.

327. The *centre of momentum* of a system in motion, is the place of the resultant of the momenta, of all the particles of the system. An adequate force opposed to this resultant, will destroy the whole momentum of the system at once; that is, the momenta of all the parts of the system, on different sides of this centre, will be extinguished in the same instant, and no rotatory effect will result. Hence, if the system rotate about a fixed axis, and the motion be stopped by a force opposed at the centre of momentum, no pressure is produced on the axis of motion.

The centre of momentum, is commonly called the centre of percussion. Or rather, the centre of momentum and that of percussion coincide.

The *centre of percussion* is that point in which an obstacle must be applied, in order to receive the whole effect of the rotatory momentum. This is evidently a definition of one of the properties of the centre of momentum : and so is that of the centre of oscillation.

*For the methods of determining the centres of oscillation and percussion, see Vince's Fluxions.*

328. The *spontaneous centre of conversion* is a point so related, by position, to a system that combines a rotatory with a progressive motion, that the incipient motion appears to take place round it as a quiescent centre, in the same manner that a pendulous system moves with respect to a fixed axis. It is that point in the plane of rotation, which is the last in assuming a relative motion. The spontaneous centre of conversion is therefore conceived to remain for an instant at rest, when the motion is beginning ; and may be considered as the momentary centre of the motion, compounded of the rotatory and progressive motions of the system.

329. If two equal bodies or particles are invariably connected by a right line, and moving with equal velocities, and if an obstacle be opposed at some point which is not their centre of gravity, the bodies will rotate about that point ; and the velocities with which they deviate from their former directions, (or the velocities of rotation) will be as their distances from the point at which the obstacle acts.

*Fig. 84.* Let A and B be the bodies, and F the obstacle acting at P. The previous momenta of A and B are now unequal moving forces, because they act at unequal distances from P, (by the property of the lever). The more remote body (A) will advance, and the other (B) will of necessity recede. The arcs A *m*, B *n*, are as the radii A P, B P, and are described uniformly in the same time. Consequently these arcs are as the velocities. Therefore, the velocities of A and B are to each other as the distances A P, B P.

#### PROP. 144.

330. If the equal bodies A and B be at rest, and if the obstacle F, in motion, act at P, the rotatory velocities will be





the same as in (329); this is evident. And the progressive velocities of A and B will be inversely as their distances from point P.

For, to prevent the effects of F in producing progressive motion, by applying forces to A and B, in the opposite direction, these forces must be inversely as the distances (by Dynamics).

*Fig. 85.* Let  $f$  and  $f$  be the forces opposed to F; then  $f =$  the action of F on A, and  $f =$  the action of F on B. The masses being equal, the spaces described in the same time will be as the forces. Let  $m n$  be the position and altitude of A B at the end of any small moment of time. Then  $A m : B n :: f : f :: B P : A P$ . And the velocities are as the spaces described  $\therefore$  vel. A : vel. B :: B P : A P.

321. If F be an instantaneous force, and act alone at any point P, on either side of the centre of gravity, the centre of continued rotation will coincide with the centre of gravity; and this point will describe a right line (216).

#### PROP. 145.

331. To find the spontaneous centre of conversion of two equal bodies or particles connected by an invariable right line, impelled at any point of that line.

*Fig. 85.* Draw an indefinite line B n perpendicular to A B at B; and find a fourth proportional to A P, B P, and B n, and make A m equal to it, and parallel to B n; join n, m, and produce n m until it intersects B A produced, in C. The point C will be the spontaneous centre of conversion. Since A P : B P :: B n : A m, it is plain that A m : B n is the ratio of the spaces described in an instant, or of the progressive velocities (330). And if C were a fixed axis of motion connected with A B, the ratio of the spaces described would be the same. Therefore, C is the spontaneous centre of conversion (328), or the spontaneous centre of rotation.

333. If C were a fixed axis of motion, and A B invariably connected with it, the point P would be the centre of momentum or centre of percussion to the point of suspension C; for, an opposite force equal F, acting simultaneously at P, would

wholly countervail the action of  $F$ . And if  $P$  were made the centre of motion,  $C$  would be the centre of momentum; for, the momenta of  $A$  and  $B$  are as their distances from  $P$ , since the masses are equal.

Considering these momenta as unequal forces which act different ways at the extremities of the line  $AB$ , their resultant is on the other side of the greater, and acts the same way (73), and the distances of its place from  $A$  and  $B$  are inversely as their momenta or forces, (74). Momentum of  $A$  : momentum of  $B$  ::  $BP$  :  $AP$ . But  $BP$  :  $AP$  ::  $Am$  :  $Bn$  (by similar  $\triangle$ ) ::  $AC$  :  $BC$ . Therefore,  $C$  is the place of the resultant of the momenta of  $A$  and  $B$ , or it is the centre of momentum to the point of suspension  $P$ .

#### PROP. 146.

334. The angular velocities of the several parts of the system ( $A$  and  $B$ ) and of the centre of conversion ( $C$ ) about the centre of gravity ( $G$ ) are equal.

*Fig. 85.* Let  $bc$  passing through the point  $I$  in the path of  $G$  be parallel to  $BC$ . When  $G$  is at  $I$ ,  $A$  will be at  $m$ , and  $B$  at  $n$ ; and the point  $C$  will, (by a relative motion with respect to the real motion of  $G$ ) have described an arc  $cC = GI$ , and in the same instant in which  $GI$  has been described. If  $cC$  be an indefinitely small arc, it will sensibly coincide with the tangent  $CL$ , and it is described with a relative motion in a retrograde direction with respect to the real motion in  $GI$ , just as the motion in the nascent arc  $am$  is retrograde relatively to that in the arc  $bn$ . But, as there is a real motion of rotation performed by  $A$ , about the centre  $G$ , so the point  $C$ , or the spontaneous centre of conversion, is conceived to have a real motion about  $G$  likewise, in the same manner as if it were invariably connected with the system. The arc  $cC$  measures the angular velocity of the point  $C$ , and the vertical angles at  $I$  being equal, the angular velocities of  $A$ ,  $B$ , and  $C$  are equal.

COR. 1. Hence, while  $A$  and  $B$  describe complete circles of rotation about  $G$ , the point  $C$  describes a complete circle with the radius  $GC$ .

COR. 2. While the system performs one complete rotation, its centre of gravity describes a right line equal to the circumference of a circle whose radius is  $GC$ . For, in the first instant the arc described by  $C =$  the right line described by  $G$ , or  $cC = GI$ ; and the motions are supposed to continue uniform or to re-





main uninfluenced by other forces. Consequently, the circular arc described by C and the rectilinear space described by G in any given time are equal; and while C performs one revolution, G will pass over a space equal the circumference  $c$  C D  $c$ .

**COR. 3.** The ratio of the rotative velocity of any part of the system to the velocity of its centre of gravity = the ratio of its distance from (G) the centre or axis of rotation to the distance of (C) the spontaneous centre of conversion from the same axis. That is, the rotative velocity of A : velocity G (= rotative velocity of C) :: A G : C G. For, the circumferences of circles are as their radii, and A and C describe their circles of rotation in the same time. Now circ. A : circ. C :: rotative vel. A : vel. G. Hence rotative vel. A : vel. G :: A G : C G.

**PROP. 147.**

**335.** The real path of the spontaneous centre of conversion is a common cycloid; that of every particle of the system, between this point and the axis of rotation (or centre of gravity), is a trochoid; and the path of every particle, more remote than the spontaneous centre of conversion from the axis of rotation, is a curtate cycloid.

Let G H be the rectilinear path of the centre of gravity. Draw an indefinite line C L, parallel to G H. While C describes the circle C D  $c$ , the centre G will describe a right line equal to its circumference (prop. 145, cor. 2). The circle C D  $c$  may therefore be supposed to roll along C L. Then, C will describe a common cycloid; any particle between C and G, connected with the axis, will describe a trochoid; and any particle at a distance from G greater than G C, and connected with the axis, will describe a curtate cycloid. (307).

*Note.*—Suppose the common centre of inertia of the moon and earth to describe a right line, while they revolve around it with constant radii. Then, there is a point C situated in the direction of a right line joining these bodies, at such a distance from G their centre of inertia, that, if that distance (C G) be made radius of a circle, G will describe a right line equal to its circumference during one complete revolution. And, the point C will describe a cycloid, and the earth and moon trochoids, the path of the former being less inflected or curved, than that of the latter.

Suppose the centre of inertia of the earth and moon to describe a circle round the sun, and suppose in the same plane, an interior concentric circle at the distance C G: also suppose these

planets to revolve round G with constant radii. Then, the remote point C will describe a circle round G in every lunar revolution, with the radius G C; and this circle whose centre is G, may be considered as rolling on the interior circle which is concentric with the orbit of G. Hence the point C will describe an epicycloid, while the earth and moon describe epitrochoids.

Nevertheless, the radii which join the earth and moon with their centre of inertia are so small compared with the distance G C, that the orbits of the earth and moon, will always be concave towards the sun; and this is actually so, notwithstanding the deviations of G from the circular path, and the disturbances to which the motions of those bodies are subjected.

If the centre of the earth or any other planet be supposed to describe a circle around the sun, and its axis be supposed perpendicular to the plane of its orbit, any point on its surface, except the extremities of its axis, would describe epitrochoids; but the progressive motion is so great, compared with the rotatory motion, that the contrary flexure would be inconsiderable, even when the path is that of a point situated in the equator; and the path of a point near the poles would always be concave towards the sun.

#### PROP. 148.

336. The *rotatory energy* of any part of a system is as the quantity of matter of that part multiplied by the square of its velocity, or by the square of its distance from the centre or axis of motion.

That is, if  $e$  = the energy or rotatory power,

$q$  = the number of particles in the part,

$v$  = the rotative velocity,

$d$  = the distance of the part from the centre of

rotation; then  $e \doteq q v^2$ , or  $e \doteq q d^2$ .

*Fig. 86.* Let C be a fixed centre, about which the system A B or B A  $a$  is supposed to rotate; and let B C = 2 A C, C  $a$  = A C, and A = B. Then the rotatory power of B in overcoming any resistance at  $a$  will be quadruple that of A. For, if a force  $f = r$  the resistance be applied at A, C is the place of the resultant, and  $f + r$  is the force acting on  $s$  the support: and if the force  $f = \frac{1}{2} f$ , be applied at B (instead of  $f$  at A) C is the place of the resultant, and  $f + r$  is the force acting on  $s$ .

Since the effect of  $f$  acting at B = effect of  $2f$  at A; the effect of any force F at B = 2 F at A.





Now, suppose  $r$  to be the mere inertia of the system, and the forces acting at A and B to be the rotatory momenta of the equal masses A and B; these forces will be as  $A \times A m$  and  $B \times B n$ , or as  $A \times A C$  and  $B \times B C$ . Since  $B C = 2 A C$ , or  $B n = 2 A m$ , and the masses equal, the moving force of B is double that of A. But, half the moving force  $F$ , acting at B, would produce the same rotatory effect (or the same effect at  $r$ ), as the whole force  $F$ , acting at A. Therefore, the double force ( $B \times B n$ ) acting at double the distance ( $B C$ ) will produce quadruple the effect in overcoming *any resistance*.

Therefore,  $e = q v^2$ , or  $= q d^2$ .

337. That point into which, if all the matter of a rotating system were collected, it would retain the same quantity of rotatory energy, is called the *centre of gyration*. This centre is otherwise defined to be, that point into which, if all the matter of the system were condensed, the same angular velocity would be generated as in the natural state of the system, by a given force acting at a given point.

PROP. 149.

338. The distance of the centre of gyration of two equal points from the axis of rotation, is equal to the square root of half the sum of the squares of their separate distances.

Let the distances of the points from the axis be  $a$  and  $b$ ; and since the points are equal, the whole rotatory energy will be  $a^2 + b^2$ , which is equal to the sum of the particles multiplied by the square of  $\sqrt{\frac{a a + b b}{2}}$ .

PROP. 150.

339. The distance of the centre of gyration of a right line from an axis at its extremity is to its length, as 1 to  $\sqrt{3}$ .

The fluxion of the rotatory power is  $x^2 \dot{x}$ ; consequently the whole rotatory power  $\frac{1}{3} x^3$ , which is equivalent to the effect of  $x$  at the distance of  $\sqrt{\frac{1}{3}} x$ .

*Note.*—The methods of determining the position of the centre of gyration of regular figures, and systems, may be found in any

comprehensive treatise on fluxions. In practical mechanics, it is often desirable to know the point at which we may suppose the whole rotatory energy of a machine, to be concentrated; for then the quantity of the energy could be readily estimated, and the force necessary to be applied at the impelled point in order to overcome the resistance, and produce the desired velocity, would be easily ascertained.

### ERRATA.

Page 12, line 3.—For, "parallel to A," read, "parallel to A C."  
 18; 17.—For, " $\pm AB \times BP$ ," read, " $\pm 2 AB \times BP$ ."

22, 15 and 16.—For  $\times$ , read,  $+$ .

32, —The symbols of velocity and time (article 102), should be accented, so as to express the increments.

49, 21.—To "are described in equal times," add, (in the same period) "with uniform motions produced by successive impulses."

53, 17.—Corrected at the end of Part II.

57, 18.—For, "but the base D E, D e, &c." read, "but the bases D E, D e, being equal, and ultimately coinciding with the tangents to the circle and the ellipse at D, the triangular sectors standing on them are to each other as S D to C D, which measure their altitudes; that is,  $\triangle S D E : S D e :: S D (= A C) : C D$ . But (by conic sections) the area of the circle : area of the ellipse :: A C (or S D) : C D; hence,  $\triangle S D E : S D e ::$  area of the circle : area of the ellipse; and the areas of the circular and elliptic sectors are similar portions of those figures. Therefore, (since in either the circle or the ellipse, equal areas are described in equal times, by 152) the times of describing the sectors S D E, S D e, are similar portions of the whole times of revolution; and the sectors being described in the same time, the whole revolutions will be performed in the same time."

Page 102, note 2.—For, "to a distance very remote from its surface," read, "its parts very remote from each other."

Page 132, line 5.—For "fluxion," read, "fluxion."

### ADVERTISEMENT.

Elementary principles of Hydrodynamics, Pneumatics, Acoustics, and Optics, it is intended, shall be added at a future season.











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